Senior Project

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1 Abstract

In this paper, Kepler's three Laws of Planetary Motion are proven using Newton's Law of Universal Gravitation. In addition, several results pertaining to the orbital period of a satellite are derived. An equation for the velocity of a satellite, as well as the minimum and maximum velocities necessary for a satellite to stay in orbit are also derived. Finally, the anomalous orbit of Mercury is examined using Newton's Law of Universal Gravitation and Einstein's Theory of Relativity. This section assumes a basic familiarity with General Relativity though a knowledge of tensor calculus is not required to follow the analysis of Mercury's orbit.

2 Introduction

For the past two millennia, people have endeavored to mathematically model the motions of the planets. The goal of predicting these motions began with the Greeks around 4 BC. Over the course of the last 2000 years, the model of the universe has changed from geocentric to heliocentric to a universe in which neither space nor time are constant and our Solar System is of little consequence (Linton, 1).

The first significant challenge to the 1500 year old geocentric model of the universe occured in 1543 when Nicholas Copernicus published <u>On the Revolutions of the Heavenly Spheres</u> (Linton, 119). Thus began a period of rapid development in mathematical astronomy. At the beginning of the 17th century, Johannes Kepler, using Tycho Brahe's observational data and the Copernican model of a heliocentric solar system, derived his laws of planetary motion (Linton, 177). These three laws are:

(1) A planet revolves in an elliptic path with the sun as one of the foci of the ellipse

(2) The radius vector from the sun to a planet sweeps out equal areas in equal intervals of time.

(3) The squares of the periods of revolution of the planets around the sun are proportional to the cubes of their mean distances from the sun.

The next significant development in mathematical astronomy was Issac Newton's formulation of a law of attraction governing massive objects. Newton's masterpiece, the *Mathematical Principles of Natural Philosophy*, known as the *Principia* was published in three volumes over a nearly 40 year period. In 1687, Newton published the first volume which contained his three laws of motion:

(1) Every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by forces impressed upon it. (Inertia)

(2) A change is proportional to the motive force impressed and takes place along the straight line in which that force is impressed. ($F = ma$)

(3) To any action there is always an opposite and equal reaction; in other words, the actions of two bodies upon each other are always equal and always opposite in direction. (Conservation of Motion) (Linton, 263).

In 1726, Newton published the third book of his *Principia*, which contained his statement of the law of universal gravitation and the resulting motion of planetary bodies in the solar system. Newton's Law of Universal Gravitation states

$$F = \frac{GMm}{r^2}$$

where G is the gravitational constant, M and m are the mass of each body, and r is the distance between the centers of each mass. From this force law and his laws of motion, Newton was able to prove Kepler's laws of planetary motion (Linton, 272).

Over the next 150 years, Newton's Law of Universal Gravitation was shown, with few exceptions, to adequately model the motions of the planetary bodies in our solar system. Cheif among the problems with Newton's Theory of Gravity, was the discrepancy between the predicted and observed motion of Mecury. By 1850, tables of the motion of Mercury were still shockingly inaccurate relative to the tables of the other planets, and the anomolous motion of Mercury became a major problem in astronomy.

During this time, Urbain Jean Joseph Le Verrier began to work on the problem of Mercury's orbit. It was not until 1859 that Le Verrier was able to correctly model the orbit of Mercury, although he could not explain the anomaly. Le Verrier discovered that the error in predictions of Mercury's orbit was a result of the advance of its perihelion, the closest point on the orbit to the Sun. By analyzing the effect of the other planets on the perihelion advance of Mercury, Le Verrier discovered that the advance of the perihelion predicted by Newton's Theory of Gravity was less than the observed perihelion advance, although he was unable to explain the cause of this anomaly (Roseveare, 20-37).

Though Le Verrier had successfully corrected the tables of Mercury's orbit, explaining the cause of this inconsistency would take another 50 years and a new Theory of Gravity. In developing his Theories of Special and General Relativity, Einstein was not immediately concerned with solving the anomalies of planetary motion. However, after developing the Theory of General Relativity, Einstein realized that it explained the anomalous advance of Mercury's perihelion. This observational verification of General Relativity, as well as gravitational lensing of light and gravitational redshift of light, proved valuable in cementing General Relativity as a viable theory. In correcting the difference between the predicted and observed advance of Mercury's perihelion, Einstein brought to a close nearly 100 years of mathematical quarreling over the cause of Mercury's anomalous perihelion and changed our conception of the universe (Roseveare, 147-186).

In this paper, we replicate the work of Newton in proving Kepler's three Laws of Planetary Motion using Newton's Law of Universal Gravitation. In addition, we will also verify other minor results involving satellite orbit. We also examine the differences in planetary orbits as predicted by Newton's Law of Universal Gravitation and Einstein's Theory of Relativity.

3 Newton's Law of Universal Gravitation

As previously mentioned, Newton's Law of Universal Gravitation states that the force governing two massive bodies is inversely proportional to the square of the distance between their centers of mass and directly proportional to the product of their masses. In mathematical terms, we have

$$F = \frac{GMm}{r^2} \tag{1}$$

where G is the gravitational constant, M and m are the masses of each body, and r is the distance between the centers of each mass.

3.1 Kepler's Laws

For the purpose of verifying Kepler's Laws, we will find it useful to choose our reference frame such that one of the objects is stationary. We choose the larger mass M to be stationary and the smaller mass m to be in orbit around the larger mass. In Cartesian coordinates, the position of mass m is given by (x, y). The gravitational force is directed towards the origin, and has magnitude,

$$|\vec{F}| = \frac{GMm}{x^2 + y^2} = \frac{GMm}{r^2}. \tag{2}$$

3.1.1 Dynamic Equations of Motion

We have that the gravitational force is given by

$$\vec{F} = -\frac{GMm}{r^2}\hat{r} = -\frac{GMm}{r^3}\vec{r},$$

where $\hat{r}$ is the unit vector in the direction of $\vec{r}$. If we resolve the force into the x and y directions, we have

$$F_x = -\frac{GMm}{r^3}x \text{ and} \tag{3a}$$

$$F_y = -\frac{GMm}{r^3}y. \tag{3b}$$

We apply Newton's second law ($F = ma$) to each component, obtaining the dynamic equations of motion for the satellite,

$$m\frac{d^2x}{dt^2} = -\frac{GMm}{r^3}x \text{ and} \tag{4a}$$

$$m\frac{d^2y}{dt^2} = -\frac{GMm}{r^3}y. \tag{4b}$$

We cancel the mass of the satellite in both equations to obtain

$$\frac{d^2x}{dt^2} = -\frac{GM}{r^3}x \text{ and} \tag{5a}$$

$$\frac{d^2y}{dt^2} = -\frac{GM}{r^3}y. \tag{5b}$$

3.1.2 Kepler's First Law

We now prove that the shape of a closed orbit of a satellite around a planet is elliptic. From equations 5a and 5b, we derive a formula for angular momentum. We will show that this formula is equal to a constant, which implies that angular momentum is conserved through a planet's orbit. We multiply equation 5a by y and 5b by x and subtract 5b from 5a yielding

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = \frac{d}{dt}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = \frac{GM}{r^3}xy - \frac{GM}{r^3}xy = 0. \tag{6a}$$

Integrating with respect to t yields

$$x\frac{dy}{dt} - y\frac{dx}{dt} = H \tag{6b}$$

for some constant H. This result is the conservation of angular momentum where H is the conserved value of angular momentum per unit mass of the orbiting motion. We now multiply 5a by $\frac{dx}{dt}$ and 5b by $\frac{dy}{dt}$ and add them together which yields

$$\frac{dx}{dt}\left(\frac{d^2x}{dt^2}\right) + \frac{dy}{dt}\left(\frac{d^2y}{dt^2}\right) = \frac{dx}{dt}\left(-\frac{GM}{r^3}x\right) + \frac{dy}{dt}\left(-\frac{GM}{r^3}y\right).$$

We can now rearrange terms on the right side of the equation

$$\frac{dx}{dt}\left(\frac{d^2x}{dt^2}\right) + \frac{dy}{dt}\left(\frac{d^2y}{dt^2}\right) = -\frac{GM}{r^3}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

For simplicity, in the right side of the equation, we can convert to polar coordinates, such that $x = r\cos\theta$ and $y = r\sin\theta$. Likewise, $dx = \cos\theta dr - r\sin\theta d\theta$ and $dy = \sin\theta dr + r\cos\theta d\theta$. Therefore, we have

$$\begin{aligned}\frac{dx}{dt}\left(\frac{d^2x}{dt^2}\right) + \frac{dy}{dt}\left(\frac{d^2y}{dt^2}\right) &= -\frac{GM}{r^3}\left[r\cos\theta\left(\cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt}\right)\right.\\ &\qquad \left. + r\sin\theta\left(\sin\theta\frac{dr}{dt} + r\cos\theta\frac{d\theta}{dt}\right)\right]\\ &= -\frac{GM}{r^3}\left[r\cos^2\theta\frac{dr}{dt} + r\sin^2\theta\frac{dr}{dt}\right].\end{aligned}$$

This simplifies to

$$\frac{dx}{dt}\left(\frac{d^2x}{dt^2}\right) + \frac{dy}{dt}\left(\frac{d^2y}{dt^2}\right) = -\frac{GM}{r^2}\frac{dr}{dt} \tag{7a}$$

Next, we rearrange the left side of Equation 7a, yielding

$$\frac{dx}{dt}\left(\frac{d}{dt}\frac{dx}{dt}\right) + \frac{dy}{dt}\left(\frac{d}{dt}\frac{dy}{dt}\right) = -\frac{GM}{r^2}\frac{dr}{dt}.$$

We recognize that the left side of the equation is the derivative of the sum of the squares of the first derivatives of x and y. Using the Chain Rule, we find we need a factor of $\frac{1}{2}$, and we now have

$$\frac{1}{2}\frac{d}{dt}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] = -\frac{GM}{r^2}\frac{dr}{dt}.$$

We can now integrate with respect to time, which produces

$$\frac{1}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] - \frac{GM}{r} = E, \tag{7b}$$

for some constant E. We can see that the first terms of equation 7b is a statement of kinetic energy per unit mass. In addition, the second term is a statement of potential energy per unit mass. Thus the constant E represents the total energy per unit mass. Thus total energy is conserved in the system. Because there are no external forces on the two body system, conservation of energy is expected.

For convenience, we now convert equations 6b and 7b to polar coordinates. The transformation of coordinates is given by

$$x = r\cos\theta \text{ and } y = r\sin\theta \tag{8a}$$

with the derivatives given by

$$dx = \cos\theta dr - r\sin\theta d\theta \text{ and } dy = \sin\theta dr + r\cos\theta. \tag{8b}$$

We begin with equation 6b, we substitute for x and y, and rearrange terms. We then have

$$
\begin{aligned}
r\cos\theta\left(\sin\theta\frac{dr}{dt}+r\cos\theta\frac{d\theta}{dt}\right)-r\sin\theta\left(\cos\theta\frac{dr}{dt}-r\sin\theta\frac{d\theta}{dt}\right) &= H \\
r^2\cos^2\theta\frac{d\theta}{dt}+r^2\sin^2\theta\frac{d\theta}{dt} &= H \\
r^2\frac{d\theta}{dt} &= H. \qquad (9a)
\end{aligned}
$$

Next we substitute for x and y in equation 7b which yields

$$
\begin{aligned}
\frac{1}{2}\left[\left(\cos\theta\frac{dr}{dt}-r\sin\theta\frac{d\theta}{dt}\right)^2+\left(\sin\theta\frac{dr}{dt}+r\cos\theta\frac{d\theta}{dt}\right)^2\right]-\frac{GM}{r} &= E \\
\frac{1}{2}\left[\cos^2\theta\left(\frac{dr}{dt}\right)^2-2r\cos\theta\sin\theta\frac{dr}{dt}\frac{d\theta}{dt}+r^2\sin^2\theta\left(\frac{d\theta}{dt}\right)^2\right] & \\
+\frac{1}{2}\left[\sin^2\theta\left(\frac{dr}{dt}\right)^2+2r\cos\theta\sin\theta\frac{dr}{dt}\frac{d\theta}{dt}+r^2\cos^2\theta\left(\frac{d\theta}{dt}\right)^2\right]-\frac{GM}{r} &= E \\
\frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2\right]-\frac{GM}{r} &= E. \qquad (9b)
\end{aligned}
$$

We then take the initial position of the satellite to be r_0 and the initial velocity to be v_0. We also define the angle between the initial position vector and the initial velocity vector to be α.

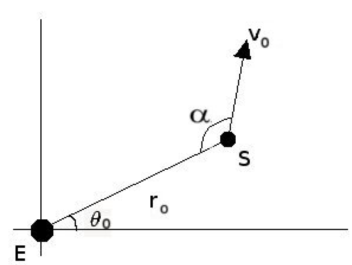

Figure 1: Initial Conditions of Launching (Kwok)

Thus, for conservation of angular momentum, we have

$$H = r_0^2\left(\frac{d\theta}{dt}\right)_{t=0} = r_0 v_0 \sin\alpha \qquad (10a)$$

and for conservation of energy, we have

$$E = \frac{v_0^2}{2}-\frac{GM}{r_0}. \qquad (10b)$$

Substituting these values into equations 9a and 9b, we find

$$r^2 \frac{d\theta}{dt} = r_0 v_0 \sin\alpha, \text{ and} \tag{11a}$$

$$\left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right). \tag{11b}$$

By eliminating dependence on t by using $\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}$, equation 11b can be transformed to

$$\left(\frac{dr}{d\theta}\right)^2 \left(\frac{d\theta}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2 = v_0^2 + \frac{2GM}{r} - \frac{2GM}{r_0}.$$

We substitute Equation 11a for $\frac{d\theta}{dt}$ which yields

$$\left(\frac{dr}{d\theta}\right)^2 \left(\frac{r_0 v_0 \sin\alpha}{r^2}\right)^2 + r^2 \left(\frac{r_0 v_0 \sin\alpha}{r^2}\right)^2 = v_0^2 + \frac{2GM}{r} - \frac{2GM}{r_0}.$$

We then rearrange terms and solve for $\left(\frac{dr}{d\theta}\right)^2$ to get.

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^4}{r_0^2 v_0^2 \sin^2\alpha}\left(v_0^2 - \frac{2GM}{r_0} + \frac{2GM}{r} - \frac{r_0^2 v_0^2 \sin^2\alpha}{r^2}\right).$$

We take the square root of both sides and simplify to solve for $\frac{dr}{d\theta}$ giving

$$\frac{dr}{d\theta} = \frac{r^2}{r_0 v_0 \sin\alpha}\sqrt{v_0^2 - \frac{2GM}{r_0} + \frac{2GM}{r} - \frac{r_0^2 v_0^2 \sin^2\alpha}{r^2}}. \tag{12a}$$

We now have a first order separable differential equation. We can begin the arduous process of solving for θ in terms of r. It is advantageous to substitute $r = \frac{1}{z}$ and thus $dr = -\frac{dz}{z^2}$. We then have

$$-\frac{1}{z^2}\frac{dz}{d\theta} = \frac{1}{v_0 r_0 z^2 \sin\alpha}\sqrt{\left(v_0^2 - \frac{2GM}{r_0}\right) + 2GMz - v_0^2 r_0^2 \sin^2\alpha z^2}.$$

Solving for $d\theta$ yields

$$-d\theta = \frac{v_0 r_0 \sin\alpha dz}{\sqrt{\left(v_0^2 - \frac{2GM}{r_0}\right) + 2GMz - v_0^2 r_0^2 \sin^2\alpha z^2}}. \tag{12b}$$

We now complete the square in the denominator in order to substitute a trigonometric function before integrating. The complete derivation of this is included in Appendix A, but can be ignored for continuity. From completing the square in the denominator of equation 12b, we have

$$-d\theta = \frac{v_0 r_0 \sin\alpha dz}{\sqrt{\frac{(G^2M^2 - 2GMv_0^2 r_0 \sin^2\alpha + v_0^4 r_0^2 \sin^2\alpha)}{v_0^2 r_0^2 \sin^2\alpha} - \left(v_0 r_0 \sin\alpha z - \frac{GM}{v_0 r_0 \sin\alpha}\right)^2}}. \quad \text{(12c)}$$

We rearrange terms and find a suitable trigonometric substitution in order to integrate. We rewrite equation 12c as

$$-d\theta = \frac{v_0 r_0 \sin\alpha dz}{\sqrt{\frac{(G^2M^2 - 2GMv_0^2 r_0 \sin^2\alpha + v_0^4 r_0^2 \sin^2\alpha)}{v_0^2 r_0^2 \sin^2\alpha}}\sqrt{1 - \left(\frac{v_0^2 r_0^2 \sin^2\alpha}{G^2M^2 - 2GMv_0^2 r_0 \sin^2\alpha + v_0^4 r_0^2 \sin^2\alpha}\right)\left(v_0 r_0 \sin\alpha z - \frac{GM}{v_0 r_0 \sin\alpha}\right)^2}}.$$

We use the trigonometric substitution

$$\frac{v_0 r_0 \sin\alpha}{\sqrt{G^2M^2 - 2GMv_0^2 r_0 \sin^2\alpha + v_0^4 r_0^2 \sin^2\alpha}}\left(v_0 r_0 \sin\alpha z - \frac{GM}{v_0 r_0 \sin\alpha}\right) = \cos U$$

and consequently,

$$dz = -\frac{\sqrt{G^2M^2 - 2GMv_0^2 r_0 \sin^2\alpha + v_0^4 r_0^2 \sin^2\alpha}}{v_0^2 r_0^2 \sin^2\alpha}\sin(U)dU.$$

We then substitute into equation 12c, which yields

$$d\theta = \frac{\sin(U)dU}{\sqrt{1 - \cos^2(U)}} = dU.$$

Integrating from θ_0 to θ yields

$$U = \theta - \theta_0.$$

We now reverse substitute for U. We know

$$U = \arccos\left(\frac{v_0 r_0 \sin\alpha}{\sqrt{G^2M^2 - 2GMv_0^2 r_0 \sin^2\alpha + v_0^4 r_0^2 \sin^2\alpha}}\left(v_0 r_0 \sin\alpha z - \frac{GM}{v_0 r_0 \sin\alpha}\right)\right).$$

Taking the cosine of both sides yields

$$\cos(\theta - \theta_0) = \frac{v_0^2 r_0^2 \sin^2\alpha}{\sqrt{G^2M^2 - 2GMv_0^2 r_0 \sin^2\alpha + v_0^4 r_0^2 \sin^2\alpha}}\left(z - \frac{GM}{v_0^2 r_0^2 \sin^2\alpha}\right).$$

We then solve for z. Rearranging terms yields

$$v_0 r_0 \sin\alpha z = \frac{GM}{v_0 r_0 \sin\alpha} + \sqrt{v_0^2 - \frac{2GM}{r_0} + \frac{G^2M^2}{v_0^2 r_0^2 \sin^2\alpha}}\cos(\theta - \theta_0).$$

We substitute $z = 1/r$ and solve for r yielding

$$r = \frac{v_0 r_0 \sin\alpha}{\frac{GM}{v_0 r_0 \sin\alpha} + \sqrt{v_0^2 - \frac{2GM}{r_0} + \frac{G^2M^2}{v_0^2 r_0^2 \sin^2\alpha}}\cos(\theta - \theta_0)}.$$

Finally, we can simplify and find

$$r = \frac{v_0^2 r_0^2 \sin^2\alpha / GM}{1 + \frac{v_0 r_0 \sin\alpha}{GM}\sqrt{v_0^2 - \frac{2GM}{r_0} + \frac{G^2M^2}{v_0^2 r_0^2 \sin^2\alpha}}\cos(\theta - \theta_0)}. \tag{13}$$

We now have the orbital path in polar form, r as a function of θ. To understand this orbital path geometrically, we would like to associate the solution with the equation of a conic section. In polar form, the equation of a conic section is

$$r = \frac{pe}{1 + e\cos\theta}, \tag{14}$$

where e is the eccentricity of the conic and p is the distance from the focus to the directrix. The conic is an ellipse when $e < 1$, a parabola when $e = 1$, and a hyperbola when $e > 1$.

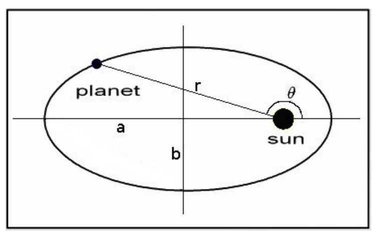

Figure 2: Elliptical Orbit of a Planet about the Sun

Comparing equations 13 and 14 and taking $\theta_0 = 0$ (the geometric interpretation of this is that the initial position vector lies along the x-axis), we find that

$$e = \sqrt{1 - \frac{v_0^2 r_0^2 \sin^2\alpha}{G^2M^2}\left(\frac{2GM}{r_0} - v_0^2\right)}, \tag{15a}$$

$$pe = \frac{v_0^2 r_0^2 \sin^2\alpha}{G^2M^2} \tag{15b}$$

We now have that the solution of the orbital path of a satellite is a conic section with the type of orbit (i.e. elliptic, parabolic, or hyperbolic) governed by the eccentricity e. Of these orbits, the only closed orbit is elliptic, and thus we have Kepler's First Law: a planet revolves in elliptic path with the sun as one of the foci of the ellipse.

3.1.3 Kepler's Second Law

We now prove Kepler's Second Law, which states that the radius vector from the sun to a planet sweeps out equal areas in equal intervals of time. As we seen in Figure 1, the physical manifestation of this property is that the satellite has the greatest velocity when it is closest to the central mass and least velocity when it is furthest from the central mass.

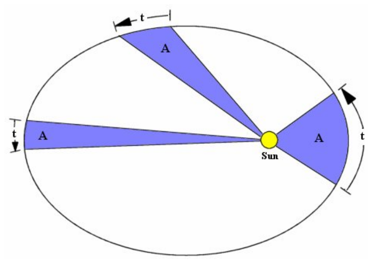

Figure 3: Kepler's Second Law (Wikipedia)

We have already defined $\vec{r}$ as the vector from the central mass to the satellite and additionally we say $r = |\vec{r}|$. We now let $d\vec{r}$ be the vector tangent to the orbit over which the satellite moves in time dt. Thus,

$$dA = \frac{1}{2}|\vec{r} \times d\vec{r}|.$$

or equivalently

$$dA = \frac{1}{2}|\vec{r} \times \vec{r}\frac{d\theta}{dt}dt|.$$

We simplify to get

$$\frac{dA}{dt} = \frac{1}{2}|\vec{r} \times \vec{r}|\frac{d\theta}{dt} = \frac{1}{2}|\vec{r}||\vec{r}|\frac{d\theta}{dt}.$$

Thus,

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}.$$

But from Equation 9a, we know

$$r^2\frac{d\theta}{dt} = H.$$

Thus,

$$\frac{dA}{dt} = \frac{H}{2}$$

and as a satellite moves around an object, the position vector $\vec{r}$ sweeps out equal areas in equal times. In addition, we see that Kepler's Second Law is simply another way of describing conservation of angular momentum.

3.1.4 Kepler's Third Law

Finally, we prove Kepler's Third Law, which states the squares of the periods of revolution of the planets around the sun are proportional to the cubes of their mean distances from the sun. Using Kepler's Second Law and equation 10a, we have

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{H}{2} = \frac{1}{2}r_0 v_0 \sin\alpha. \tag{16}$$

In Section 2.2, we showed that the path of a revolving satellite is elliptic. We also know that the area of an ellipse is πab, where a and b are the lengths of the semi-major axis and semi-minor axis respectively. To find the period of revolution T, we integrate equation 16 from $t = 0$ to $t = T$ yielding

$$\begin{aligned} \pi ab = A(T) - A(0) &= \int_0^T \frac{dA}{dt}dt \\ &= \int_0^T \frac{1}{2}r_0 v_0 \sin\alpha dt \\ &= \frac{1}{2}r_0 v_0 \sin\alpha T. \end{aligned} \tag{17}$$

From our initial conditions for deriving the orbital path and Equation 15b, we have that $r_{\min} = pe/(1+e)$ occurs when $\theta = 0$ and the maximum value $r_{\max} = pe/(1-e)$ occurs when $\theta = \pi$. Therefore the length of the semi-major axis, a is

$$a = \frac{\max + \min}{2} = \frac{pe}{1-e^2} = \frac{GM}{\frac{2GM}{r_0} - v_0^2} \tag{18a}$$

and similarly, the length of the semi-major axis is

$$b = \frac{\max - \min}{2} = a\sqrt{1-e^2} = \frac{v_0 r_0 \sin\alpha}{\sqrt{\frac{2GM}{r_0} - v_0^2}}. \tag{18b}$$

We can now substitute equations 18a and 18b into equation 17, and obtain the period of revolution of the elliptic path of a orbiting satellite,

$$T = \frac{2\pi GM}{\left(\frac{2GM}{r_0} - v^2\right)^{3/2}} = \frac{2\pi a^{3/2}}{\sqrt{GM}} \tag{19}$$

or, to match the wording of the Third Law,

$$T^2 = \frac{4\pi^2 a^3}{GM}.$$

Thus we have Kepler's Third Law: the square of the period of revolution of a planet around the sun is proportional to the cube of the mean distance from the sun.

3.1.5 Orbital Period

We now examine the orbital period of a satellite. We wish to express the time t in terms of the angular displacement θ. We cover the major steps of this derivation here, while the complete derivation is included in Appendix B. We begin with the statement of conservation of angular momentum in polar coordinates using the initial conditions. We have

$$r^2 \frac{d\theta}{dt} = r_0 v_0 \sin\alpha.$$

We then use the equation for position r in terms of θ,

$$r = \frac{v_0^2 r_0^2 \sin^2\alpha / GM}{1 + \frac{v_0 r_0 \sin\alpha}{GM}\sqrt{v_0^2 - \frac{2GM}{r_0} + \frac{G^2M^2}{v_0^2 r_0^2 \sin^2\alpha}}\cos(\theta - \theta_0)}.$$

Substituting for r and rearranging terms yields

$$dt = \frac{v_0^3 r_0^3 \sin^3\alpha d\theta}{G^2M^2\left(1 + \sqrt{1 - \frac{v_0^2 r_0^2 \sin^2\alpha}{G^2M^2}\left(\frac{2GM}{r_0} - v_0^2\right)}\cos(\theta - \theta_0)\right)^2}.$$

We integrate from 0 to θ and take $\theta_0 = 0$, and $\alpha = 0$, meaning the initial position is at the perihelion. We have

$$t = \int_0^\theta \frac{v_0^3 r_0^3 d\theta}{G^2M^2\left(1 + \sqrt{1 - \frac{v_0^2 r_0^2 \sin^2\alpha}{G^2M^2}\left(\frac{2GM}{r_0} - v_0^2\right)}\cos(\theta)\right)^2}.$$

From our definition of eccentricity, e, we have

$$t = \int_0^\theta \frac{v_0^3 r_0^3 d\theta}{G^2M^2\left(1 + e\cos(\theta)\right)^2}.$$

In order to evaluate this integral, We begin by multiplying by $\frac{(1-e^2)}{(1-e^2)}$ and our integral becomes

$$t = \frac{v_0^3 r_0^3}{G^2M^2(1-e^2)}\int_0^\theta \frac{(1-e^2)d\theta}{\left(1 + e\cos(\theta)\right)^2}.$$

We now utilize the Weierstrass Substitution

$$U = \tan\left(\frac{\theta}{2}\right).$$

Using properties of trigonometric functions, we find

$$\cos\theta = \frac{1-U^2}{1+U^2} \text{ and } \sin\theta = \frac{2U}{1+U^2}.$$

We also find

$$d\theta = \frac{2}{1+U^2}dU.$$

We substitute these into our integral to get

$$\frac{v_0^3 r_0^3}{G^2M^2(1-e^2)}\int \frac{2(1-e^2)dU}{\left[1+e\left(\frac{1-U^2}{1+U^2}\right)\right]^2(1+U^2)}.$$

This integral is equivalent to

$$\frac{v_0^3 r_0^3}{G^2M^2(1-e^2)}\int \frac{-2e(1+e)+2e(1-e)U^2}{[(1+e)+(1-e)U^2]^2}dU+\frac{v_0^3 r_0^3}{G^2M^2(1-e^2)}\int \frac{2}{(1+e)+(1-e)U^2}dU.$$

We integrate the first term by reversing the Weierstass substitution and subsequently using integration by parts. The second term is integrated by using another trigonometric substitution. We are then left with

$$t = \frac{v_0^3 r_0^3}{G^2M^2}\int_0^\theta \frac{d\theta}{(1+e\cos(\theta))^2}$$

$$= \frac{v_0^3 r_0^3}{G^2M^2(1-e^2)}\left[\frac{-e\sin\theta}{1+e\cos\theta}+\frac{2}{\sqrt{1-e^2}}\tan^{-1}\left[\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right]+nT\right],$$

where T is the period of the orbit, given by

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM}}$$

and n is the number of complete revolutions the planet has made, or for $\phi > 2\pi$, $\phi = \theta + 2\pi n$. Thus, the nT term accounts for the planet moving through more than one complete orbit. As previously mentioned, the complete derivation of this equation is presented in Appendix B.

3.2 The Cosmic Velocities

Now that we have proven Kepler's Laws, we would like to explore some other results governing orbits. We begin by finding an equation for the velocity of a satellite at any position r. In our orbital system, we know that the total energy E, given in the equation,

$$\frac{1}{2}\left[\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2\right]-\frac{GM}{r}=E,$$

is constant by the principle of conservation of energy. The first term of this equation is the kinetic energy of the system, and the second term is the potential energy. We can now show that the type of orbit is dependent on the total energy of the system. We begin by recognizing that the square of the velocity is equal to the sum of the squares of the radial and angular components of the velocity. We have

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = v^2,$$

or equivalently,

$$v^2 = \frac{2GM}{r} + 2E.$$

We then recall equation 15a

$$e = \sqrt{1 - \frac{v_0^2 r_0^2 \sin^2\alpha}{G^2M^2}\left(\frac{2GM}{r_0} - v_0^2\right)}.$$

Substituting our equation for velocity into the previous equation, we have

$$e = \sqrt{1 + \frac{2v_0^2 r_0^2 \sin^2\alpha}{G^2M^2}E}.$$

We can now compare the three conditions, $E = 0$, $E < 0$, and $E > 0$. When $E = 0$, we have $e = 1$, and thus the orbit is parabolic. The condition in which $E = 0$ means that the kinetic energy is equal to the potential energy, which in physical terms means that the object has the minimum amount of energy necessary to escape orbit. Because every term in the coefficient of E is squared, we know the coefficient is positive. Thus, for $E < 0$, we have $e < 1$ and thus the orbit is elliptic. The object does not have sufficient kinetic energy to escape orbit. Similarly, for $E > 0$, we have $e > 1$ and therefore the orbit is hyperbolic. Because the kinetic energy is greater than the potential energy, the object escapes a closed orbit.

We now move on to orbital speed. We have previously established that

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right).$$

This is equivalent to

$$v^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

which gives velocity in terms of the initial conditions and the current position. We have also previously shown that

$$a = \frac{GM}{\frac{2GM}{r_0} - v_0^2}.$$

Rearranging terms yields,

$$v_0^2 = \frac{2GM}{r_0} - \frac{GM}{a}.$$

Substituting this statement of our initial conditions into our equation for velocity in terms of r yields

$$v^2 = \frac{2GM}{r_0} - \frac{GM}{a} + \frac{2GM}{r} - \frac{2GM}{r_0} = GM\left(\frac{2}{r} - \frac{1}{a}\right),$$

or equivalently,

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}.$$

Thus we have an expression for the velocity of a satellite at any position r. We can also see that if $a = r$, a circular orbit, that

$$v^2 = \frac{GM}{r}.$$

We now demonstrate that the initial propulsion speed v_0 of a satellite launched from the Earth's surface has to fall within a certain range in order for the satellite to remain in orbit, which is to say, neither hit the surface of the earth, nor move out of the earth's gravitational field. We know from the definition of conic sections, that the size of e in the equation 14 determines the shape of the orbit. We can see from equation 15b that the sign of quantity $(v_0^2 - 2GM/r_0)$ determines whether e is less than, equal to, or greater than 1. We can then define v_0^* as the velocity when $e = 1$, or

$$v_0^* = \sqrt{\frac{2GM}{r_0}} = \sqrt{\frac{2gR^2}{r_0}} = \sqrt{\frac{R}{r_0}}\sqrt{2gR}, \tag{20}$$

where $g = \frac{GM}{R^2}$. From equation 15a we can see that when the velocity is less than the minimum initial velocity, $v < v_0^*$, then $e < 1$ and the object will either fall back to Earth or orbit elliptically. If $v = v_0^*$, then $e = 1$ and the object will follow a parabolic path, having just enough energy to escape the Earth's gravitational field. Finally, if $v > v_0^*$, then $e > 1$ and the orbit will be hyperbolic, with the object also escaping the Earth's gravitational field.

We now know that v_0 must be less than $\sqrt{\frac{R}{r_0}}\sqrt{2gR}$ in order for a satellite to stay within the gravitational field of the earth. We derive the minimum velocity necessary for the satellite to stay in orbit. Thus, we are limited by $r_{\min} = r_0$, the point at which the satellite collides with the surface of the earth. We know $r_{\min} = pe/(1+e)$ and, using equations 15a and 15b, we have

$$r_0 = \frac{1}{1+e}\frac{v_0^2 r_0^2 \sin^2\alpha}{GM}, \tag{21a}$$

which is equivalent to

$$\frac{v_0^2 r_0^2 \sin^2\alpha}{GM} - 1 = \sqrt{1 - \frac{v_0^2 r_0^2 \sin^2\alpha}{G^2M^2}\left(\frac{2GM}{r_0} - v_0^2\right)}. \tag{21b}$$

The left hand side of equation 21b is always nonnegative, so we have

$$v_0 \geq \sqrt{\frac{GM}{r_0}} \csc\alpha \geq \sqrt{\frac{GM}{r_0}} = \sqrt{\frac{R}{r_0}}\sqrt{gR}. \tag{22}$$

Thus we have that the initial speed v_0 of a projectile launched from the surface of the earth must fall within the range,

$$\sqrt{\frac{R}{r_0}}\sqrt{gR} \leq v_0 \leq \sqrt{\frac{R}{r_0}}\sqrt{2gR}, \quad r_0 > R. \tag{23}$$

Finally, if we square both sides of equation 21b, we find that $\sin\alpha = 1$ and thus $\alpha = \pi/2$. This means that at $r_{\min}$, the angle between the position vector and the velocity vector is $\pi/2$. Thus, the point where the satellite is closest to the earth occurs along the major axis. If the satellite is launched with minimum velocity from this point, the satellite will be further from the earth than $r_{\min}$ for all other points along its orbit.

We now examine an alternative method for deriving the second cosmic velocity, that is the maximum initial velocity an object can be launched from the surface of a planet and remain in orbit. We previously established that for an object launched from the planet's surface to remain in orbit,

$$\sqrt{gR} < v_0 < \sqrt{2gR}.$$

The alternative approach is to find the minimum velocity such that the object will leave the planet and never return, which is to say that as r increases without bound, the velocity remains nonnegative.

We know from $F = ma$, that

$$\frac{d^2y(t)}{dt^2} = -\frac{GM}{r^2}.$$

If we define $r = R + y(t)$ such that R is the radius of the planet and $y(t)$ is the altitude of the object above the surface of the planet, then our previous equation becomes

$$\frac{d^2y(t)}{dt^2} = -\frac{R^2g}{[R+y(t)]^2}. \tag{24a}$$

We can now rearrange terms and solve for the velocity of the object in terms of its height above the surface of the planet. We can rewrite equation 24a as

$$\frac{d[v(t)]}{dt} = -\frac{R^2g}{[R+y(t)]^2}. \tag{24b}$$

Rearranging terms and multiplying each side of equation 24b by v yields

$$vdv = -R^2g \cdot \frac{vdt}{[R+y(t)]^2}. \tag{25a}$$

We can now use the substitution $U = R + y(t)$ and thus $dU = vdt$ to get

$$vdv = -R^2 g \cdot \frac{dU}{U^2}, \tag{25b}$$

or

$$\int_{v_0}^{v} vdv = \int_{R}^{y(t)-R} -R^2 g \cdot \frac{dU}{U^2}. \tag{25c}$$

Evaluating this integral, we have

$$\frac{1}{2}(v^2 - v_0^2) = R^2 g \frac{1}{U} = R^2 g \frac{1}{R + y(t)}\bigg|_0^{y(t)}. \tag{26a}$$

This simplifies to

$$v(t)^2 = v_0^2 + 2R^2 g \left[\frac{1}{R + y(t)} - \frac{1}{R}\right]. \tag{26b}$$

As previously stated, the minimum velocity such that the object never returns to the planet occurs when the velocity reaches zero as $y(t) \to \infty$. In equation 26b, if we let $y(t) \to \infty$, we have

$$v_\infty^2 = v_0^2 + 2R^2 g \left[0 - \frac{1}{R}\right],$$

where v_∞ is the velocity when $y(t)$ is arbitrarily large. We simplify the previous equation to

$$v_\infty^2 = v_0^2 - 2Rg.$$

If $v_\infty > 0$ at this point, the object will leave earth. Thus we have

$$0 < v_0^2 - 2Rg,$$

or

$$v_0 > \sqrt{2Rg}.$$

Thus we have verified that the second cosmic velocity is $v_0 = \sqrt{2Rg}$.

4 Einstein's Theory of General Relativity

4.1 The Anomalous Advance of Mercury's Perihelion

For nearly two hundred years after Newton's *Principia* was published, the inverse square law was the accepted method by which to model the motions of the planets and their satellites. Newton's Law of Universal Gravitation was even used by Urbain Jean Joseph Le Verrier to predict the existence of Neptune. By the mid 19th century, however, problems began to arise, most notably with predictions of the orbit of Mercury. As Le Verrier began his work on the orbit of Mercury, the errors in the prediction of Mercury's orbit were so large that there was nearly an hour between the predicted and actual start of transit, the time at which Mercury passes in front of the sun and so can be clearly seen and an accurate position recorded. By 1845, Le Verrier had reduced this error to 16 seconds, which was still unacceptable in comparison to the predicted orbits of other planets (Linton, 437).

Le Verrier discovered that the error in predictions of Mercury's orbit was a result of the advance of its perihelion, the closest point on the orbit to the Sun. By analyzing the effect of the other planets on the perihelion advance, Le Verrier discovered that Mercury's perihelion advanced by 565" per century rather than the 527" predicted by Newtonian mechanics. Despite this discovery, Le Verrier was unable to explain the cause of the anomalous 38". (Roseveare, 20-37).

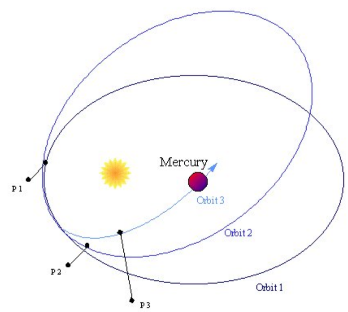

Figure 4: Perihelion Advance (Cornell)

Though the problem of Mercury's orbit was problematic for Newton's Law of Universal Gravitation, the scientific community was not yet willing to question the dogma of Newtonian mechanics. Instead other causes of Mercury's perihelion advance were investigated. Based on the success of Le Verrier's prediction of the existence and position of Neptune, similar techniques were employed in hopes of discovering a planet between the sun and mercury that could be effecting the orbit. This hypothetical planet, deemed Vulcan, was "observed" many times during the second half of the 19th century, though each discovery was subsequently discredited (Linton, 441). Alternative hypotheses included that the sun was oblate, as well as effects from the ether, the medium through which light was thought to travel.

Though models of Mercury's orbit were now mathematically accurate, determining the cause of the anomaly would require a shift in the way we conceived of space and time. Shortly after the start of the 20th century, Einstein had already published his theory of special relativity, which held that "the laws of physics are of the same form in all inertial frames, and that in any given inertial frame, the speed of light is the same whether the source is at rest or in uniform motion" (Linton, 454). In response, the mathematician Hermann Minkowski translated special relativity into a geometric framework, and as such created the concept of four-dimensional spacetime. Within special relativity, which is to say in the absence of a massive object, spacetime, as a four-dimensional Euclidean space, can be thought of as 'flat.' Any object moving through this flat spacetime takes the shortest path and thus travels in a straight line. In developing general relativity, Einstein postulated that mass had the effect of deforming Minkowskian spacetime. An object moving through this deformed space time will still take the shortest path possible, leading to the elliptical orbit of a planet (Linton, 463). The two-dimensional analog of this can be seen in Figure 3, mass deforms flat space time and causes the trajectory of the moving mass to curve.

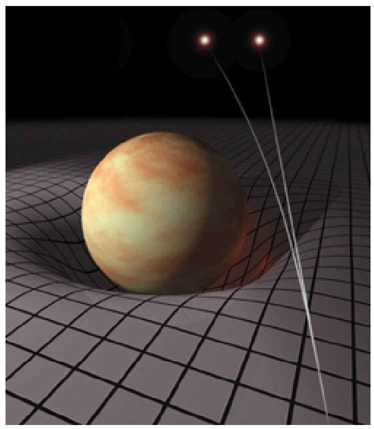

Figure 5: Mass deforms space time (Carroll)

Unfortunately the mathematical underpinnings of these concepts require a knowledge of tensor calculus and are thus beyond the scope of this paper. Rather we will begin with the general relativistic analog of the classical angular momentum and energy equation (previously defined as 11b) derived by simplifying the spherically symmetric Swcharzchild solution to Einstein's field equations.

After developing his gravitational theory, Einstein soon realized that it explained the anomalous advance of Mercury's perihelion. In correcting the difference between the predicted and observed advance of Mercury's perihelion, Einstein brought to a close nearly 100 years of mathematical quarreling over the cause of Mercury's anomalous perihelion and changed our conception of the universe (Roseveare, 147-186).

4.2 Relativistic Angular Momentum and Energy

In order to investigate the effect of General Relativity on the orbit of Mercury, we must investigate the general-relativity analog of the previously stated classical angular momentum and energy equation given by

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2 = v_0^2 + 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right). \tag{11b}$$

We let E be a constant related to the energy of the orbit and given by

$$E = \frac{v_0^2}{2} - \frac{GM}{r_0}.$$

Substituting this into Equation 11b yields

$$\left(\frac{dr}{d\phi}\right)^2 \left(\frac{d\phi}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt}\right)^2 = 2Eh^2 + \frac{2GM}{r}. \tag{27a}$$

We also have h, the angular momentum per unit mass given by

$$h = r^2 \left(\frac{d\phi}{dt}\right).$$

Dividing by h^2, we have

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{r^2} = \frac{2E}{h^2} + \frac{2GM/r}{r^4\left(\frac{d\phi}{dt}\right)^2}. \tag{27b}$$

Because energy and angular momentum are conserved, $\frac{2E}{h^2}$ is a constant which, for simplicity, we will call E_0 and thus we have

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{r^2} = E_0 + \frac{2GM/r}{r^4\left(\frac{d\phi}{dt}\right)^2}. \tag{27c}$$

We then introduce the substitution $u = \frac{1}{r}$, and thus $du = -\frac{dr}{r^2}$. This yields

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = E_0 + \frac{2GM}{h^2}u. \tag{27d}$$

This is the classical angular momentum and energy equation. In solving Einstein's field equations, which is beyond the scope of this project, a general relativistic analog of the previous equation is derived. We find that the general relativistic equation is the classical equation with another term, equal to $2GMu^3/c^2$. We expect that this term will change the orbit of a satellite from pure Keplerian motion. We then have the equation

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = E_0 + \frac{2GM}{h^2}u + \frac{2GMu^3}{c^2}. \tag{28a}$$

The quantity $2GM/c^2$ is small compared to the radius of planetary orbits. We can denote this quantity as ε, and we have

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = E_0 + \frac{2GM}{h^2}u + \varepsilon u^3. \tag{28b}$$

Aphelion and Perihelion occur where $du/d\phi = 0$, and thus we have

$$\varepsilon u^3 - u^2 + \left(\frac{2GM}{h^2}\right) u + E_0 = 0. \tag{29}$$

This cubic equation then has three roots, let them be denoted u_1, u_2, and u_3. Because ε is small, unless u is large, the roots will be close to the roots of the analogous classical equation

$$u^2 - \left(\frac{2GM}{h^2}\right) u - E_0 = 0.$$

Thus, we say u_1 and u_2 are close to the roots of the Newtonian model and thus we let u_1 be the aphelion and u_2 be the perihelion. Thus we know $u_1 \leq u \leq u_2$. We also know that $u_1 + u_2 + u_3 = 1/\varepsilon$. For a proof of this see appendix C. Because $1/\varepsilon$ is large, u_3 is also large and thus has no physical meaning in our model. Substituting our solution for equation 29 into equation 28a, we have

$$\frac{du}{d\phi} = [\varepsilon(u - u_1)(u_2 - u)(u_3 - u)]^{\frac{1}{2}}. \tag{30a}$$

Rearranging terms, we have

$$\frac{du}{d\phi} = [[(u - u_1)(u_2 - u)(\varepsilon(u_3 - u))]^{\frac{1}{2}}.$$

We then rewrite this as

$$\frac{du}{d\phi} = [[(u - u_1)(u_2 - u)(\varepsilon(u_1 - u_1 + u_2 - u_2 + u_3 - u))]^{\frac{1}{2}}.$$

We then group terms such that

$$\frac{du}{d\phi} = [[(u - u_1)(u_2 - u)(\varepsilon(u_1 + u_2 + u_3) - \varepsilon(u_1 + u_2 + u)))]^{\frac{1}{2}}.$$

We can cancel ε and $u_1 + u_2 + u_3$. This yields

$$\frac{du}{d\phi} = [[(u - u_1)(u_2 - u)(1 - \varepsilon(u_1 + u_2 + u)))]^{\frac{1}{2}}.$$

In terms of $d\phi/du$, we have

$$\frac{d\phi}{du} = \frac{1}{[(u - u_1)(u_2 - u)]^{1/2}}[1 - \varepsilon(u_1 + u_2 + u)]^{-1/2}. \tag{30b}$$

Using the first order approximation from the Taylor expansion, $(1 + x)^k = 1 + kx$, we have

$$\frac{d\phi}{du} \approx \frac{1 + \frac{1}{2}\varepsilon(u_1 + u_2 + u)}{[(u - u_1)(u_2 - u)]^{1/2}}. \tag{30c}$$

We now let $\alpha = \frac{1}{2}(u_1 + u_2)$ and $\beta = \frac{1}{2}(u_2 - u_1)$. We then have

$$\frac{d\phi}{du} = \frac{1 + \frac{1}{2}\varepsilon(2\alpha + u)}{[-u^2 + (u_1 + u_2)u - u_1 u_2]^{1/2}}.$$

This is equivalent to

$$\frac{d\phi}{du} = \frac{1 + \frac{1}{2}\varepsilon(2\alpha + u)}{[\frac{1}{4}(u_2^2 - 2u_1 u_2 + u_1^2) - u^2 + (u_1 + u_2)u - \frac{1}{4}(u_1^2 + 2u_1 u_2 + u_2^2)]^{1/2}}.$$

This reduces to

$$\frac{d\phi}{du} = \frac{1 + \frac{1}{2}\varepsilon(2\alpha + u)}{[\beta^2 - (u - \alpha)^2]^{1/2}}. \tag{31a}$$

Integrating equation 31 with respect to u from u_1 to u_2 allows us to find the angle between an aphelion and the subsequent perihelion. We have

$$\Delta\phi = \int_{u_1}^{u_2} \frac{1 + \frac{1}{2}\varepsilon(2\alpha + u)}{[\beta^2 - (u - \alpha)^2]^{1/2}} du. \tag{31b}$$

Rearranging terms, we have

$$\Delta\phi = \int_{u_1}^{u_2} \frac{\frac{1}{2}\varepsilon(u - \alpha) + 1 + \frac{3}{2}\varepsilon\alpha}{[\beta^2 - (u - \alpha)^2]^{1/2}} du.$$

We can split the integral into two terms such that,

$$\Delta\phi = \frac{1}{2}\varepsilon \int_{u_1}^{u_2} \frac{(u - \alpha)}{[\beta^2 - (u - \alpha)^2]^{1/2}} du + \int_{u_1}^{u_2} \frac{1 + \frac{3}{2}\varepsilon\alpha}{[\beta^2 - (u - \alpha)^2]^{1/2}} du.$$

The first term is relatively easy to integrate and we simply have

$$\frac{1}{2}\varepsilon \int_{u_1}^{u_2} \frac{(u - \alpha)}{[\beta^2 - (u - \alpha)^2]^{1/2}} du = -\frac{1}{2}\varepsilon(\beta^2 - (u - \alpha)^2)^{1/2}|_{u_1}^{u_2}.$$

This yields

$$\frac{1}{2}\varepsilon \int_{u_1}^{u_2} \frac{(u - \alpha)}{[\beta^2 - (u - \alpha)^2]^{1/2}} du = -\frac{1}{2}\varepsilon\left(\frac{(u_2 - u_1)^2}{4} - (u_2 - \frac{1}{2}(u_2 + u_1))\right)^2 + \frac{1}{2}\varepsilon\left(\frac{(u_2 - u_1)^2}{4} - (u_1 - \frac{1}{2}(u_2 + u_1))\right)^2.$$

This simplifies to

$$\frac{1}{2}\varepsilon \int_{u_1}^{u_2} \frac{(u - \alpha)}{[\beta^2 - (u - \alpha)^2]^{1/2}} du = 0.$$

For the second term, we have

$$\int_{u_1}^{u_2} \frac{1 + \frac{3}{2}\varepsilon\alpha}{[\beta^2 - (u - \alpha)^2]^{1/2}} du = \left[(1 + \frac{3}{2}\varepsilon\alpha) \arcsin\left(\frac{u - \alpha}{\beta}\right)\right]_{u_1}^{u_2} = (1 + \frac{3}{2}\varepsilon\alpha)\pi.$$

Doubling $\Delta\phi$ and subtracting 2π gives the angle between successive perihelions. We then have that each orbit advances the perihelion by

$$\phi = 3\varepsilon\alpha\pi = \frac{3GM\pi}{c^2}(u_1 + u_2) = \frac{3GM\pi}{c^2}\left(\frac{1}{r_1} + \frac{1}{r_2}\right), \tag{32}$$

where r_1 and r_2 are the values of r at aphelion and perihelion. In the case of Mercury, r_1 and r_2 are small compared to orbital radii of the other planets. As a result, Mercury has a larger perihelion advance then the other planets.. In addition, because the orbit of Mercury is more elliptical than the orbits of most of the other planets, aphelion and perihelion are easier to observe. For Mercury, the advancement of perihelion works out to about 43" per century. While small, it is enough to be observed, and previous to Einstein's Theory of General Relativity, was unexplainable.

5 Conclusion

In this paper, we began by verifying Kepler's Laws of Planetary Motion using Newton's Law of Universal Gravitation. We then derived several results pertaining to the orbital period of a satellite, including solving for orbital period in terms of angular displacement. We then derived an equation for the velocity of a satellite, as well as the cosmic velocities, the minimum and maximum velocities necessary for a satellite to stay in orbit. Finally, we examined the anomalous orbit of Mercury using Newton's Law of Universal Gravitation and Einstein's Theory of Relativity.

In our use of classical mechanics in exploring planetary motion, we assumed several things. First, we assumed that we have a two-body system, using the example of a satellite orbiting Earth. In reality, a massive body's orbit is perturbed by the gravitational attraction to any other mass in its vicinity. Thus, the orbits of the planets are not simply two-body systems with a given planet orbiting the Sun, rather the orbit of each planet is affected by the other planets. Unfortunately, this multiple-body problem is very complicated and must be modeled numerically. Fortunately, the Sun is vastly more massive than the planets and thus the two-body model is a reasonable approximation. Our second assumption was that the Earth is is a homogeneous sphere rather than a heterogeneous ellipsoid. The ellipsoidal shape of the Earth causes small perturbations of the elliptic path of a satellite.

Additional considerations include that the perihelion advance of Mercury's orbit is not simply due to the effects of General Relativity. The gravitational effects of other planets, as well as the fact that we are observing Mercury from a moving platform, contribute to Mercury's perihelion advance. In our analysis we simply derived the additional orbital advance that was discovered by Le Verrier and explained by General Relativity.

Our analysis also assumes no knowledge of tensor calculus in the derivation of Mercury's perihelion advance from the relativistic angular momentum and energy equation. Given an understanding of tensor calculus, the angular momentum and energy equation can be derived from Einstein's field equations.

Reasonable areas for further work would be deriving additional results pertaining to the orbits of individual satellites as well as analysis of the three-body problem. A useful resource for additional work in this area is Solar System Dynamics by Carl D. Murray and Stanley F. Dermott. Given an understanding of tensor calculus, the angular momentum and energy equation could also be derived, as well as other results in General Relativity. A useful resource for additional work in this area is A Short Course in General Relativity as referenced in the bibliography. Further information on the historical context of the mathematics covered in the paper can be found in From Eudoxus to Einstein: A History of Mathematical Astronomy as referenced in the bibliography.

6 Appendices

6.1 Appendix A

On page 7, we used the standard process of completing the square to transform equation 12b into equation 12c. The full derivation is given here. We begin with equation 12b

$$-d\theta = \frac{v_0 r_0 \sin\alpha dz}{\sqrt{\left(v_0^2 - \frac{2GM}{r_0}\right) + 2GMz - v_0^2 r_0^2 \sin^2\alpha z^2}}. \tag{12b}$$

The denominator is

$$(v_0^2 r_0^2 \sin^2\alpha) z^2 - (2GM)z + \left(\frac{2GM}{r_0} - v_0^2\right).$$

This is equivalent to

$$\left(v_0 r_0 \sin\alpha z - \frac{GM}{v_0 r_0 \sin\alpha}\right)^2 + C,$$

where C is some constant that we will determine. We then expand the above statement, yielding

$$(v_0^2 r_0^2 \sin^2\alpha) z^2 - (2GM)z + \frac{G^2 M^2}{v_0^2 r_0^2 \sin^2\alpha} + C.$$

Equating this statement with the square of the denominator of equation 12b and simplifying we have

$$\frac{G^2 M^2}{v_0^2 r_0^2 \sin^2\alpha} + C = \frac{2GM}{r_0} - v_0^2.$$

Solving for C, we have

$$C = \frac{2GM}{r_0} - v_0^2 - \frac{G^2 M^2}{v_0^2 r_0^2 \sin^2\alpha}.$$

We then find a common denominator, which gives us

$$C = \frac{2GMv_0^2 r_0 \sin^2\alpha - v_0^4 r_0^2 \sin^2\alpha - G^2 M^2}{v_0^2 r_0^2 \sin^2\alpha}.$$

Thus equation 12b becomes

$$-d\theta = \frac{v_0 r_0 \sin\alpha dz}{\sqrt{\frac{(G^2M^2 - 2GMv_0^2 r_0 \sin^2\alpha + v_0^4 r_0^2 \sin^2\alpha)}{v_0^2 r_0^2 \sin^2\alpha} - \left(v_0 r_0 \sin\alpha z - \frac{GM}{v_0 r_0 \sin\alpha}\right)^2}}. \tag{12c}$$

6.2 Appendix B

In section 2.1.5, we examined the orbital period of a satellite and found an equation for the time t in terms of the angular displacement θ. The complete derivation is presented here. We begin with the statement of conservation of angular momentum in polar coordinates using the initial conditions. We have

$$r^2 \frac{d\theta}{dt} = r_0 v_0 \sin\alpha.$$

We then use the equation for position r in terms of θ,

$$r = \frac{v_0^2 r_0^2 \sin^2\alpha / GM}{1 + \frac{v_0 r_0 \sin\alpha}{GM}\sqrt{v_0^2 - \frac{2GM}{r_0} + \frac{G^2M^2}{v_0^2 r_0^2 \sin^2\alpha}}\cos(\theta - \theta_0)}.$$

Substituting for r yields

$$\left[\frac{v_0^4 r_0^4 \sin^4\alpha / G^2M^2}{\left(1 + \frac{v_0 r_0 \sin\alpha}{GM}\sqrt{v_0^2 - \frac{2GM}{r_0} + \frac{G^2M^2}{v_0^2 r_0^2 \sin^2\alpha}}\cos(\theta - \theta_0)\right)^2}\right]\frac{d\theta}{dt} = r_0 v_0 \sin\alpha.$$

Rearranging terms, we have

$$dt = \frac{v_0^3 r_0^3 \sin^3\alpha d\theta}{G^2M^2\left(1 + \frac{v_0 r_0 \sin\alpha}{GM}\sqrt{v_0^2 - \frac{2GM}{r_0} + \frac{G^2M^2}{v_0^2 r_0^2 \sin^2\alpha}}\cos(\theta - \theta_0)\right)^2}.$$

Again we rearrange terms, which yields

$$dt = \frac{v_0^3 r_0^3 \sin^3\alpha d\theta}{G^2M^2\left(1 + \sqrt{\frac{v_0^4 r_0^2 \sin^2\alpha}{G^2M^2} - \frac{2v_0^2 r_0 \sin^2\alpha}{GM} + 1}\cos(\theta - \theta_0)\right)^2}.$$

This is equivalent to

$$dt = \frac{v_0^3 r_0^3 \sin^3\alpha d\theta}{G^2M^2\left(1 + \sqrt{1 - \frac{v_0^2 r_0^2 \sin^2\alpha}{G^2M^2}\left(\frac{2GM}{r_0} - v_0^2\right)}\cos(\theta - \theta_0)\right)^2}.$$

For $\theta_0 = 0$, and $\alpha = 0$, meaning the initial position is the perihelion, we have

$$t = \int_0^\theta \frac{v_0^3 r_0^3 d\theta}{G^2M^2\left(1 + \sqrt{1 - \frac{v_0^2 r_0^2 \sin^2\alpha}{G^2M^2}\left(\frac{2GM}{r_0} - v_0^2\right)}\cos(\theta)\right)^2}.$$

From our definition of eccentricity, e, we finally have

$$t = \int_0^\theta \frac{v_0^3 r_0^3 d\theta}{G^2M^2\left(1 + e\cos(\theta)\right)^2}.$$

In order to evaluate this integral, We begin by multiplying by $\frac{(1-e^2)}{(1-e^2)}$ such that our integral becomes

$$t = \frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \int_0^{\theta} \frac{(1-e^2)d\theta}{(1+e\cos(\theta))^2}.$$

We now utilize the Weierstrass Substitution such that

$$U = \tan\left(\frac{\theta}{2}\right).$$

We can use properties of trigonometric functions to find expressions for $\cos(\theta)$, $\sin(\theta)$, and $d\theta$. We begin with $\cos\theta$ and we know that

$$U^2 = \frac{1-\cos\theta}{1+\cos\theta}.$$

We now rewrite this equation, and we have

$$\begin{aligned}
U^2 + U^2\cos\theta &= 1-\cos\theta \\
\cos\theta + U^2\cos\theta &= 1-U^2 \\
(1+U^2)\cos\theta &= 1-U^2 \\
\cos\theta &= \frac{1-U^2}{1+U^2}
\end{aligned}$$

To find an expression for $\sin\theta$, we begin with

$$\begin{aligned}
\sin^2\theta &= 1-\cos^2\theta \\
\sin^2\theta &= 1-\left(\frac{1-U^2}{1+U^2}\right)^2 \\
\sin^2\theta &= \frac{1+2U^2+U^2}{(1+U^2)^2} - \frac{1-2U^2+U^2}{(1+U^2)^2} \\
\sin^2\theta &= \frac{4U^2}{(1+U^2)^2}
\end{aligned}$$

and we have

$$\sin\theta = \frac{2U}{1+U^2}.$$

To find $d\theta$, we can use the derivative of $\sin\theta$. We then have

$$\cos\theta d\theta = \frac{(1+U^2)\cdot 2 - (2U)(2U)}{(1+U^2)^2}dU.$$

We then have

$$d\theta = \frac{(1+U^2)\cdot 2 - (2U)(2U)}{(1+U^2)^2}\frac{1+U^2}{1-U^2}dU.$$

This simplifies to

$$d\theta = \frac{2}{1+U^2}dU.$$

We can now substitute into our integral. This yields

$$\frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \int \frac{2(1-e^2)dU}{\left[1+e\left(\frac{1-U^2}{1+U^2}\right)\right]^2 (1+U^2)}.$$

Rearranging terms, we have

$$\frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \int \frac{2(1-e^2)dU}{\left[\frac{(1+U^2)+e(1-U^2)}{1+U^2}\right]^2 (1+U^2)}.$$

This simplifies to

$$\frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \int \frac{2(1-e^2)(1+U^2)dU}{[(1+e)+(1-e)U^2]^2}.$$

Expanding the numerator, we have

$$\frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \int \frac{2(1-e^2+U^2-e^2U^2)dU}{[(1+e)+(1-e)U^2]^2}.$$

We can rewrite the numerator, yielding

$$\frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \int \frac{2(1-e^2)+2(1-e^2)U^2}{[(1+e)+(1-e)U^2]^2}dU.$$

Factoring yields

$$\frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \int \frac{2(1+e)(1-e)+2(1+e)(1-e)U^2}{[(1+e)+(1-e)U^2]^2}dU.$$

We now rewrite the numerator as

$$\frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \int \frac{-2e(1+e)+2(1+e)+2e(1-e)U^2+2(1-e)U^2}{[(1+e)+(1-e)U^2]^2}dU.$$

We can now separate this into two terms

$$\frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \int \frac{-2e(1+e)+2e(1-e)U^2}{[(1+e)+(1-e)U^2]^2}dU + \frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \int \frac{2(1+e)+2(1-e)U^2}{[(1+e)+(1-e)U^2]^2}dU.$$

The second term reduces, and we have

$$\frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \int \frac{-2e(1+e)+2e(1-e)U^2}{[(1+e)+(1-e)U^2]^2}dU + \frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \int \frac{2}{(1+e)+(1-e)U^2}dU.$$

We begin by integrating the first term,

$$\frac{v_0^3 r_0^3}{G^2 M^2(1-e^2)} \int \frac{-2e(1+e)+2e(1-e)U^2}{[(1+e)+(1-e)U^2]^2} dU.$$

We rewrite this as

$$= \frac{v_0^3 r_0^3}{G^2 M^2(1-e^2)} \int \frac{-2e[(1+e)-(1-e)U^2]}{[(1+e)+(1-e)U^2]^2} \left(\frac{1+U^2}{2}\right)\left(\frac{2}{1+U^2}\right) dU.$$

Expanding the numerator, we have

$$= \frac{v_0^3 r_0^3}{G^2 M^2(1-e^2)} \int \frac{-e[(1+e)+(1+e)U^2-(1-e)U^2-(1-e)U^4]}{[(1+e)+(1-e)U^2]^2} \left(\frac{2}{1+U^2} dU\right).$$

This is equivalent to

$$= \frac{v_0^3 r_0^3}{G^2 M^2(1-e^2)} \int \frac{-e[(1+e)+U^2+2eU^2-U^2-(1-e)U^4]}{[(1+e)+(1-e)U^2]^2} \left(\frac{2}{1+U^2} dU\right).$$

Rearranging terms in the numerator yields

$$= \frac{v_0^3 r_0^3}{G^2 M^2(1-e^2)} \int \frac{-e[(1-U^4)+e-2eU^2+eU^4+4eU^2]}{[(1+e)+(1-e)U^2]^2} \left(\frac{2}{1+U^2} dU\right).$$

We now factor terms in the numerator and have

$$= \frac{v_0^3 r_0^3}{G^2 M^2(1-e^2)} \int \frac{-e(1-U^2)(1+U^2)-e^2(1-U^2)(1-U^2)-e(2U)^2}{[(1+e)+(1-e)U^2]^2} \left(\frac{2}{1+U^2} dU\right).$$

We now multiply by $\left(\frac{1+U^2}{1+U^2}\right)^2$ such that we have

$$= \frac{v_0^3 r_0^3}{G^2 M^2(1-e^2)} \int \frac{\frac{-e(1-U^2)(1+U^2)-e^2(1-U^2)(1-U^2)-e(2U)^2}{(1+U^2)^2}}{\left[\left(\frac{1+U^2}{1+U^2}\right)+e\left(\frac{1-U^2}{1+U^2}\right)\right]^2} \left(\frac{2}{1+U^2} dU\right).$$

We then rearrange terms, which yields

$$= \frac{v_0^3 r_0^3}{G^2 M^2(1-e^2)} \int \frac{-e\left(\frac{1-U^2}{1+U^2}\right)-e^2\left(\frac{1-U^2}{1+U^2}\right)^2-e^2\left(\frac{2U}{1+U^2}\right)^2}{\left[1+e\left(\frac{1-U^2}{1+U^2}\right)\right]^2} \left(\frac{2}{1+U^2} dU\right).$$

We can combine the first and second terms of the numerator such that

$$= \frac{v_0^3 r_0^3}{G^2 M^2(1-e^2)} \int \frac{-e\left(\frac{1-U^2}{1+U^2}\right)\left[1+e\left(\frac{1-U^2}{1+U^2}\right)\right]-e^2\left(\frac{2U}{1+U^2}\right)^2}{\left[1+e\left(\frac{1-U^2}{1+U^2}\right)\right]^2} \left(\frac{2}{1+U^2} dU\right).$$

We can now substitute for $\cos\theta$, $\sin\theta$, $d\theta$. This yields

$$= \frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \int \frac{(1+e\cos\theta)(-e\cos\theta)-(-e\sin\theta)(-e\sin\theta)}{(1+e\cos\theta)^2} d\theta.$$

Splitting the integral into two terms, we have

$$= \frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \left[-\int \frac{e\cos\theta}{1+e\cos\theta} d\theta - \int \frac{e^2\sin^2\theta}{(1+e\cos\theta)^2} d\theta \right].$$

We begin integration by parts on the second term such that

$$U = e\sin\theta \qquad dU = e\cos\theta d\theta$$

$$V = \tfrac{1}{1+e\cos\theta} \quad dV = \tfrac{e\sin\theta}{(1+e\cos\theta)^2} d\theta$$

We now have

$$= \frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \left[-\int \frac{e\cos\theta}{1+e\cos\theta} d\theta - \frac{e\sin\theta}{1+e\cos\theta} + \int \frac{e\cos\theta}{(1+e\cos\theta)} d\theta \right].$$

The first and third terms cancel and we are left with

$$= \frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \frac{-e\sin\theta}{1+e\cos\theta}.$$

We now integrate the second term,

$$\frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \int \frac{2}{(1+e)+(1-e)U^2} dU.$$

We multiply this by $\frac{\sqrt{1-e}}{\sqrt{1-e}}$ and $\frac{1+e}{1+e}$ such that we have

$$= \frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \int \frac{2\frac{\sqrt{1-e}}{\sqrt{1-e}} \frac{1+e}{\sqrt{1+e}\sqrt{1+e}}}{(1+e)+(1-e)U^2} dU.$$

Rearranging terms yields

$$= \frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \int \frac{\frac{2}{\sqrt{1-e}\sqrt{1+e}} \left(\frac{\sqrt{1-e}}{\sqrt{1+e}}\right)}{\left[\frac{(1+e)+(1-e)U^2}{1+e}\right]} dU.$$

This is equivalent to

$$= \frac{v_0^3 r_0^3}{G^2M^2(1-e^2)} \frac{2}{\sqrt{1-e}\sqrt{1+e}} \int \frac{\left(\frac{\sqrt{1-e}}{\sqrt{1+e}}\right)}{1+\left(\frac{\sqrt{1-e}}{\sqrt{1+e}}U\right)^2} dU.$$

We now introduce another trigonometric substitution such that

$$U = \sqrt{\frac{1+e}{1-e}}\tan(V)\ dU = \sqrt{\frac{1+e}{1-e}}\sec^2(V)dV.$$

Substituting, we have

$$= \frac{v_0^3 r_0^3}{G^2M^2(1-e^2)}\frac{2}{\sqrt{1-e}\sqrt{1+e}}\int \frac{\sec^2(V)}{1+\tan^2(V)}dV.$$

We know $\sec^2(V) = 1+\tan^2(V)$, and so we reduce and integrate yielding

$$= \frac{v_0^3 r_0^3}{G^2M^2(1-e^2)}\frac{2}{\sqrt{1-e}\sqrt{1+e}}\int dV = \frac{v_0^3 r_0^3}{G^2M^2(1-e^2)}\frac{2}{\sqrt{1-e}\sqrt{1+e}}\cdot V.$$

Substituting for V yields

$$= \frac{v_0^3 r_0^3}{G^2M^2(1-e^2)}\frac{2}{\sqrt{1-e^2}}\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}U\right).$$

Substituting for U, we have

$$= \frac{v_0^3 r_0^3}{G^2M^2(1-e^2)}\frac{2}{\sqrt{1-e^2}}\tan^{-1}\left[\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right].$$

Finally, we have

$$t = \frac{v_0^3 r_0^3}{G^2M^2}\int_0^\theta \frac{d\theta}{(1+e\cos(\theta))^2}$$

$$= \frac{v_0^3 r_0^3}{G^2M^2(1-e^2)}\left[\frac{-e\sin\theta}{1+e\cos\theta} + \frac{2}{\sqrt{1-e^2}}\tan^{-1}\left[\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right] + nT\right],$$

where T is the period of the orbit, given by

$$T = \frac{2\pi a^{3/2}}{\sqrt{GM}}$$

and n is the number of complete revolutions the planet has made, or for $\phi > 2\pi$, $\phi = \theta + 2\pi n$.

6.3 Appendix C

Proving $u_1 + u_2 + u_3 = \frac{1}{\varepsilon}$.
When we expand $\varepsilon(u-u_1)(u_2-u)(u_3-u)$, the u^2 term works out to be $-\epsilon(u_1 + u_2 + u_3)u^2$. Equating this with the u^2 term in our energy equation, we have $-\epsilon(u_1+u_2+u_3)u^2 = -u^2$, and thus

$$u_1 + u_2 + u_3 = \frac{1}{\varepsilon}.$$

7 Bibliography

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