BAIRE ONE FUNCTIONS

JOHNNY HU

Abstract. This paper gives a general overview of Baire one functions, including examples as well as several interesting properties involving bounds, uniform convergence, continuity, and F_σ sets. We conclude with a result on a characterization of Baire one functions in terms of the notion of first return recoverability, which is a topic of current research in analysis [6].

1. History

Baire one functions are named in honor of René-Louis Baire (1874-1932), a French mathematician who had research interests in continuity of functions and the idea of limits [1]. The problem of classifying the class of functions that are convergent sequences of continuous functions was first explored in 1897. According to F.A. Medvedev in his book *Scenes from the History of Real Functions*, Baire was interested in the relationship between the continuity of a function of two variables in each argument separately and its joint continuity in the two variables [2]. Baire showed that every function of two variables that is continuous in each argument separately is not pointwise continuous with respect to the two variables jointly [2]. For instance, consider the function

$$f(x,y) = \frac{xy}{x^2+y^2}.$$

Observe that for all points $(x,y) \neq 0$, f is continuous and so, of course, f is separately continuous. Next, we see that,

$$\lim_{(x,0)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{xy}{x^2+y^2} = 0$$

as (x,y) approaches $(0,0)$ along the x-axis and y-axis respectively. However, as (x,y) approaches $(0,0)$ along the line $y=x$, we have

$$\lim_{(x,x)\to(0,0)} \frac{xy}{x^2+y^2} = \frac{1}{2}.$$

Hence, f is continuous along the lines $y=0$ and $x=0$, but is discontinuous overall at $(0,0)$. It can be shown that f is a Baire one function. It was this idea that motivated the development of the classification of Baire class functions, which was later generalized in Baire's 1899 dissertation *Sur les fonctions des variables réelles* [2].

In general, a function that is *Baire class* n, is the pointwise limit of a sequence of Baire class $n-1$ functions, where Baire class 0 consists of the set of all continuous functions [3]. For instance, a function that is Baire class one (which we will define in the next section) is the pointwise limit of continuous functions. Functions that are Baire class two are the pointwise limit of a sequence of Baire class one functions, and so on.

2. Preliminaries and Definition of Baire One

Before we define what it means for a function to be Baire one, it is necessary to first introduce some background that will be helpful. We assume that the reader has some familiarity with sequences of real numbers and what it means for a sequence to converge. It is possible to make analogous statements in terms of functions. Let $\{f_n\}$ define a sequence of functions whose domain and range is the set of real numbers. We are interested to see if $\{f_n\}$ converges to a specific function at each point in its interval for which it is defined. This type of convergence is known as **pointwise convergence** and we state it below.

Definition 1 (Pointwise Convergence)**.** *Let $\{f_n\}$ be a sequence of functions defined on a set D and also let f be a function defined on D. If the sequence $\{f_n(x)\}$ converges to $f(x)$ for each $x \in D$, then $\{f_n\}$ converges* ***pointwise*** *to f on D.*

In other words, $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in I$. An example to illustrate the definition is as follows. Let $g_n : [0,1] \to \mathbb{R}$ be defined as $g_n(x) = x^n$. We see that the sequence converges pointwise to

$$g(x) = \begin{cases} 0, & \text{for } x \in [0,1); \\ 1, & \text{if } x = 1. \end{cases}$$

From Figure 1, we notice that each g_n is continuous on the closed interval $[0,1]$. Given our definition of pointwise convergence, we now state the formal definition of a Baire one function.

Definition 2. [5] *Let $D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ is called a* ***Baire one function*** *if f is the pointwise limit of a sequence of continuous functions, that is, if there is a sequence $\{f_n\}$ of functions continuous on D such that for every $x \in D$, $f(x) = \lim_{n\to\infty} f_n(x)$.*

The function g above is an example of a Baire one function since each function g_n is continuous and converges pointwise to g on $[0,1]$ as $n \to \infty$.

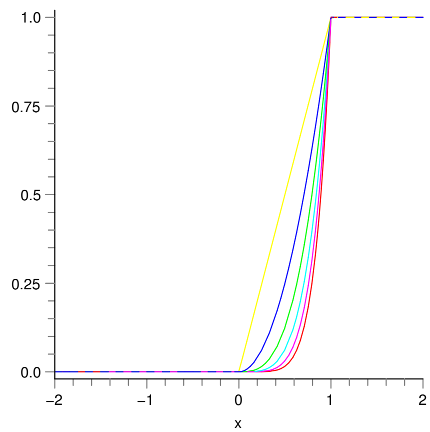

FIGURE 1. The function g_n, $n = 1, 2, 3, 4, 5, 6$

3. Examples and Some Properties of Baire One Functions

We now present some examples as well as some interesting properties of Baire one functions. As shown in the previous section, a Baire one function does not have to be continuous, however all continuous functions are Baire one by definition. To show this, suppose j is a continuous function and let $j_n : \mathbb{R} \to \mathbb{R}$ where $j_n(x) = j(x) + 1/n$, for each $n \in \mathbb{Z}^+$. It follows that $j(x) = \lim_{n\to\infty} j_n(x)$.

Another question that we might ask is whether or not we can we extend the domain of a Baire one function to the entire real line. The answer is yes. Let $k_n : \mathbb{R} \to \mathbb{R}$ be defined as the following:

$$k_n(x) = \begin{cases} 0, & \text{if } x < 5 - \frac{1}{n}; \\ nx - (5n - 1), & \text{if } 5 - \frac{1}{n} \le x \le 5; \\ -nx + (5n + 1), & \text{if } 5 < x \le 5 + \frac{1}{n}; \\ 0, & \text{if } x > 5 + \frac{1}{n}. \end{cases}$$

We see that k_n converges pointwise to k on $\mathbb{R}$ (see Figure 2) where,

$$k(x) = \begin{cases} 0, & \text{if } x \ne 5; \\ 1, & \text{if } x = 5. \end{cases}$$

This shows that functions can still be Baire one if they are discontinuous at one point. One of the questions one might ask is the following, how many points of continuity does a Baire one function have to have?

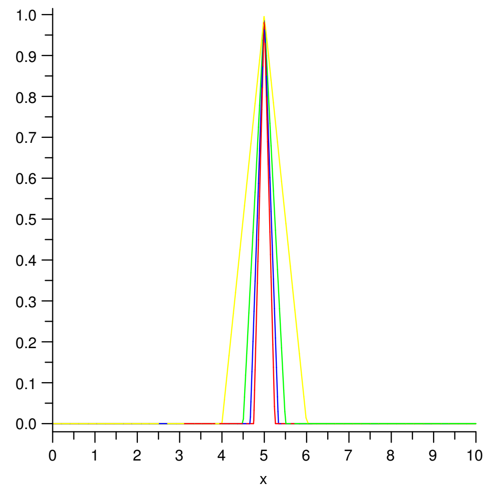

FIGURE 2. The function k_n, $n = 1, 2, 3, 4$

More specifically, can a Baire one function be everywhere discontinuous? The answer is no, and the reason will be addressed later on. In fact, we will see that Baire one functions must have many points of continuity.

However, suppose $f : \mathbb{R} \to \mathbb{R}$ has a finite number of jump discontinuities at $x = x_1, x_2, \ldots, x_k$. Is f Baire one? The answer is yes. Let $x = x_1, x_2, \ldots, x_k$ be the finite number of discontinuities of f, where $x_1 < x_2 < x_3 < \cdots < x_k$. We will show the case when $k = 1$. Given any positive integer n, we see that f is continuous on the interval $(-\infty, x_1 - 1/n)$, so define $f_n(x) = f(x)$ on this interval. Likewise, $f_n(x) = f(x)$ is continuous on the interval $(x_1 + 1/n, x_2)$, where $x = x_2$ is the next point of discontinuity. We then define our function f_n such that $f_n(x)$ is a linear function on the interval $[x_1 - 1/n, x_1]$ connecting the points $f(x_1 - 1/n)$ and $f(x_1)$ together (in order to guarantee the continuity of the f_ns) and likewise on the interval $[x_1, x_1 + 1/n]$ connecting the points $f(x_1)$ and $f(x_1 + 1/n)$ in a linear manner. As a result, f_n converges to f as n approaches infinity. The same process can be repeated for the other points $\{x_2, x_3, ..., x_k\}$ where f is discontinuous. We have just shown that functions with finitely-many points of discontinuities are Baire one.

Another interesting property is as follows. Suppose f and g are Baire one functions, both mapping $D \subseteq \mathbb{R}$ into $\mathbb{R}$. Since the sequences $\{f_n\}$ and $\{g_n\}$ converge pointwise to f and g respectively, it follows by a property of convergent sequences that $\{f_n + g_n\}$ converge to $f + g$ and hence $f + g$ is Baire one. Likewise, the sequence $\{f_n g_n\}$ converges to fg by a similar argument and fg is Baire one [4].

Finally, we note that all differentiable functions are Baire one. This should not be too surprising, since all differentiable questions are continuous, and all continuous functions are Baire one by definition. A proof of this fact is given in the next theorem.

Theorem 1. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Then f' is Baire one.*

Proof. We will show that $f'(x) = \lim_{n\to\infty} f_n(x)$. From calculus, we know that the definition of the derivative of a function f is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

We want our expression of f' to be in the form of an infinite limit. Hence, the following is an equivalent statement.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{n\to\infty} \frac{f\left(x + \frac{1}{n}\right) - f(x)}{\frac{1}{n}} \\ &= \lim_{n\to\infty} n\left[f\left(x + \frac{1}{n}\right) - f(x)\right]. \end{aligned}$$

Let $f_n(x) = n\left[f\left(x + \frac{1}{n}\right) - f(x)\right]$, and note that f_n is continuous for $n = 1, 2, 3 \ldots$. We conclude that,

$$\begin{aligned} f'(x) &= \lim_{n\to\infty} n\left[f\left(x + \frac{1}{n}\right) - f(x)\right] \\ &= \lim_{n\to\infty} f_n(x). \end{aligned}$$

$\square$

4. Bounds and Uniform Convergence

In this section, we present some properties of Baire one functions that pertain to bounds. An interesting question one might ask is whether a Baire one function $f : [a, b] \to \mathbb{R}$ must be bounded that is, does there exist a real number M such that $|f(x)| \leq M$ for all $x \in [a, b]$?

We first give an example of an unbounded Baire one function $f : [0, 1] \to \mathbb{R}$. Next, we give two properties of bounded Baire one functions and conclude with a theorem about uniform convergence.

Let f_n and f be defined as follows on the closed interval $[0, 1]$. We note that $\{f_n\}$ is a sequence of continuous functions.

$$f_n(x) = \begin{cases} n^2 x, & \text{if } 0 \leq x < \frac{1}{n}; \\ \frac{1}{x}, & \text{if } \frac{1}{n} \leq x \leq 1. \end{cases}$$

$$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ 1/x, & \text{if } 0 < x \leq 1. \end{cases}$$

It's obvious that f is unbounded on $[0, 1]$. We will show that f_n converges to f pointwise on $[0, 1]$. Let $x \in [0, 1]$. Then we have two cases to consider.

If $x = 0$, we have $f_n(0) = 0 = f(0)$ for all n.

If $0 < x \leq 1$, we note the following. Choose a positive integer N such that $1/N < x$. Then $f_n(x) = 1/x = f(x)$ for $n \geq N$, since $1/n \leq 1/N < x$. It follows that

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f(x) = f(x).$$

Hence, f_n converges pointwise to f. We have just found a Baire one function that is unbounded.

The next result shows that given a Baire one function with bound M, we can find a sequence of continuous function $\{f_n\}$ that converges pointwise to f with the same bound.

Lemma 1. *If $f : [a, b] \to \mathbb{R}$ is a bounded Baire one function with bound M, then there is a sequence $\{f_n\}$ of continuous functions such that each f_n has the same bound M and such that $\{f_n\}$ converges pointwise to f on $[a, b]$.*

Proof. Let $\{g_n\}$ be a sequence of continuous functions that converges pointwise to f on a closed interval $[a, b]$. Suppose also that f has bound M. Let $\{f_n\}$ be the sequence defined piecewise by

$$f_n(x) = \begin{cases} -M, & \text{if } g_n(x) < -M; \\ g_n(x), & \text{if } -M \le g_n(x) \le M; \\ M, & \text{if } g_n(x) > M. \end{cases}$$

It is apparent that each f_n has bound M. Since f also has bound M and each g_n converges pointwise to f on $[a, b]$, it follows that $\{f_n\}$ converges pointwise to f on $[a, b]$ with bound M. □

We use this result in the proof of the following lemma, which will in turn be used to help show the relationship between Baire one and uniform convergence.

Lemma 2. *Let $\{f_k\}$ be a sequence of Baire one functions defined on $[a, b]$ and let $\sum_{k=1}^{\infty} M_k$ be a convergent sequence of positive real numbers. If $|f_k(x)| \le M_k$ for all k and for all $x \in [a, b]$, then the function $f(x) = \sum_{k=1}^{\infty} f_k(x)$ is a Baire one function.*

Proof. Let $\{f_k\}$ be a sequence of Baire one functions defined on $[a, b]$ and let $\sum_{k=1}^{\infty} M_k$ be a convergent sequence of positive real numbers.Since each f_k is Baire one, for each positive integer k, there exists a sequence $\{g_{k_i}\}$ of continuous functions such that g_{ki} converges pointwise to f_k on $[a, b]$ and by Lemma 1, $|g_{ki}| \le M_k$ for all i. For each positive integer n, let

$$h_n = g_{1n} + g_{2n} + \cdots + g_{nn}.$$

We will show that $\{h_n\}$ converges pointwise to f on $[a, b]$. Fix a point $x \in [a, b]$ and let $\epsilon > 0$. By assumption, there exists a positive integer K such that $\sum_{k=K+1}^{\infty} M_k < \epsilon$. Choose an integer $N > K$ such that $|g_{ki}(x) - f_k(x)| < \frac{\epsilon}{K}$ for $1 \le k \le K$ and for $i \ge N$. For any $n \ge N$ we have,

$$\begin{aligned} |h_n(x) - f(x)| &= \left| \sum_{k=1}^{n} g_{nk}(x) - \sum_{k=1}^{\infty} f_k(x) \right| \\ &\le \left| \sum_{k=1}^{n} [g_{kn}(x) - f_k(x)] \right| + \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \\ &\le \sum_{k=1}^{K} |g_{kn}(x) - f_k(x)| + \sum_{k=K+1}^{n} |g_{kn}(x)| + \sum_{k=K+1}^{\infty} |f_k(x)| \\ &< \sum_{k=1}^{K} \frac{\epsilon}{K} + 2 \sum_{k=K+1}^{\infty} M_k \\ &< 3\epsilon \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, it follows that $\{h_n(x)\}$ converges pointwise to $f(x)$ for all $x \in [a, b]$. $\square$

While the definition of Baire category one functions relies simply on the fact that a sequence of continuous functions converges pointwise to a limit function f, pointwise convergence does not guarantee that the limit function necessarily retains the same properties of each of the functions f_n. In order to ensure that the functions f_n and f have the same properties, a stronger kind of convergence (defined below) is required [4].

Definition 3 (Uniform Convergence)**.** *Let $\{f_n\}$ be a sequence of functions defined on a set D and let f be defined on D. The sequence $\{f_n\}$ converges **uniformly** to f on D if for each $\epsilon > 0$ there exists a positive integer N such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in D$ and for all $n \geq N$.*

From the definition we see that uniform convergence is pointwise convergence with the added property that given any $\epsilon > 0$, the choice of N is independent of $x \in D$ [4]. Figure 3 shows that for sufficiently large N, the graph of f_n must lie between the dotted lines around the graph of f.

To give a simple example that illustrates the definition, consider the sequence $\{f_n\}$ defined by $f_n(x) = \cos(nx)/n$ for all n. We will show that this sequence converges uniformly to the function f defined by $f(x) = 0$ for all $x \in \mathbb{R}$. Let $\epsilon > 0$ and choose a positive integer N such that $N > 1/\epsilon$. Then for all $n \geq N$ and $x \in \mathbb{R}$,

$$|f_n(x) - f(x)| = \left|\frac{\cos(nx)}{n}\right| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

The inequality shows that the sequence of functions $\{f_n\}$ converges uniformly to the limit function f on $\mathbb{R}$.

Our goal is to show that the limit function f of a uniformly convergent sequence of Baire one functions $\{f_n\}$ is also Baire one.

Theorem 2. *Let $\{f_n\}$ be a sequence of Baire one functions defined on $[a, b]$. If $\{f_n\}$ converges uniformly to f on $[a, b]$, then f is a Baire one function.*

Proof. Let $\{f_n\}$ be a sequence of Baire one functions that converges uniformly to f on $[a, b]$. By definition of uniform convergence, for each k, there exists a subsequence $\{f_{n_k}\}$ such that $|f_{n_k}(x) - f(x)| < 2^{-k}$ for all $x \in [a, b]$.

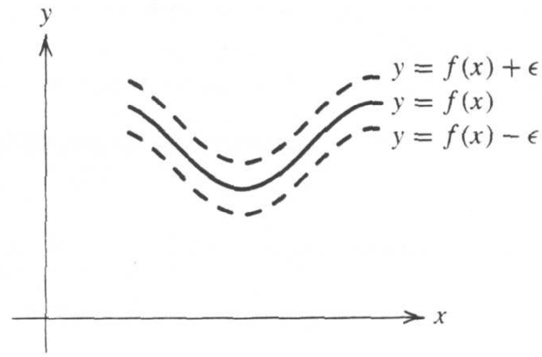

FIGURE 3. Illustration of Uniform Convergence [4]

Let's consider the sequence $\{f_{n_{k+1}} - f_{n_k}\}$. We see that by the triangle inequality,

$$\begin{aligned} |f_{n_{k+1}}(x) - f_{n_k}(x)| &\leq |f_{n_{k+1}}(x) - f(x)| + |f(x) - f_{n_k}(x)| \\ &\leq 2^{-(k+1)} + 2^{-k} \\ &= \frac{1}{2}(2^{-k}) + 2^{-k} = \left(\frac{3}{2}\right)2^{-k}. \end{aligned}$$

Let $M_k = (3/2)2^{-k}$ and note that $|f_{n_{k+1}}(x) - f_{n_k}(x)| \leq M_k$ and $\sum_{k=1}^{\infty} M_k$ is a convergent geometric series. We will show that the function $\sum_{k=1}^{\infty}[f_{n_{k+1}}(x) - f_{n_k}(x)]$ is Baire one and use the result of Lemma 2 to show that f is Baire one. Thus,

$$\begin{aligned} \sum_{k=1}^{\infty}[f_{n_{k+1}}(x) - f_{n_k}(x)] &= \lim_{N\to\infty} \sum_{k=1}^{N}[f_{n_{k+1}}(x) - f_{n_k}(x)] \\ &= \lim_{N\to\infty} \sum_{k=1}^{N} f_{n_{k+1}}(x) - \sum_{k=1}^{N} f_{n_k}(x) \\ &= f(x) - f_{n_1}(x). \end{aligned}$$

By hypothesis, the function $f_{n_1}(x)$ is Baire one. It follows by Lemma 2 that the function $\sum_{k=1}^{\infty}[f_{n_{k+1}}(x) - f_{n_k}(x)] = f(x) - f_{n_1}(x)$ is Baire one. Since $f(x) = (f(x) - f_{n_1}(x)) + f_{n_1}(x)$ and the sum of two Baire one functions is Baire one, f is also Baire one. $\square$

5. F_σ SETS

We now try to find a way to determine whether or not a function is Baire one by using properties and results of F_σ sets. We first define several properties of F_σ sets. Then, we digress briefly to present a theorem about continuous functions that is an analog of our main result of this section. Finally, we present and prove our main result.

Definition 4 (F_σ set)**.** [5] *A subset $A \subseteq \mathbb{R}$ is called an* $\mathbf{F}_\sigma$ ***set*** *if A is a countable union of closed sets. In other words, $A = \bigcup_{n=1}^{\infty} F_n$ where each F_n is a closed subset of $\mathbb{R}$.*

The following are examples of F_σ sets on the real line. Let $X = (0,1)$, $Y = [0,1)$, and $Z = [0,1]$ respectively. Then we have,

$$\begin{aligned} X &= \bigcup_{n=3}^{\infty} \left[\frac{1}{n}, 1 - \frac{1}{n}\right]; \\ Y &= \bigcup_{n=2}^{\infty} \left[0, 1 - \frac{1}{n}\right]; \\ Z &= \bigcup_{n=1}^{\infty} [0,1]; \end{aligned}$$

which are all countable unions of closed sets of $\mathbb{R}$. It's obvious from the definition that every closed set is also an F_σ set, but our constructions of X and Y show there exist sets that are F_σ and not closed. In fact, all open subsets of $\mathbb{R}$ are also F_σ [4].

We now present some properties of F_σ sets.

Lemma 3. *The intersection of two F_σ sets is an F_σ set.*

Proof. Let $A = \bigcup_{i=1}^{\infty} A_i$ and $B = \bigcup_{j=1}^{\infty} B_j$, such that A_i's and B_j's are closed sets. We see that by DeMorgan's Laws,

$$A \cap B = (\cup_{i=1}^{\infty} A_i) \cap (\cup_{j=1}^{\infty} B_j) = \cup_{i=1}^{\infty} \cup_{j=1}^{\infty} (A_i \cap B_j)$$

For each i and j, the set $A_i \cap B_j$ is a closed set. Hence $A \cap B$ is an F_σ set. □

Lemma 4. *Suppose that K and E are closed subsets of $\mathbb{R}$. Then $K - E$ is an F_σ set.*

Proof. Let K and E be closed sets. Since every closed set is an F_σ set and every open subset of $\mathbb{R}$ is also an F_σ set [4], it follows from Lemma 3 that,

$$K - E = K \cap E^C$$

is an F_σ set, where E^C denotes the complement of the set E (which is an open set). □

The next result states that a closed interval $[a, b]$ which equals a countable union of F_σ sets, can also be written as a countable union of "smaller" disjoint F_σ sets. It will be used in the proof of the main theorem of this section which gives a criterion that determines whether or not a function is Baire one.

Lemma 5. *Suppose $[a, b] = \bigcup_{k=1}^{n} A_k$ and each A_k is an F_σ set. Then $[a, b] = \bigcup_{k=1}^{n} B_k$ where each B_k is an F_σ set, $B_k \subseteq A_k$ for each k, and the B_k's are pairwise disjoint.*

Proof. Suppose $[a, b] = \bigcup_{k=1}^{n} A_k$ and each A_k is an F_σ set. By definition of F_σ sets, each set A_k is a countable union of closed sets. It follows that $[a, b] = \bigcup_{k=1}^{n} A_k = \bigcup_{i=1}^{\infty} E_i$ where each E_i is closed and $E_i \subseteq A_k$ for some k.

Next, let $H_1 = E_1$, and $H_i = E_i - (E_1 \cup \cdots \cup E_{i-1})$ for $i \geq 2$. It follows from Lemma 4 that each H_i is an F_σ set. Since $[a, b] = \bigcup_{k=1}^{n} A_k$ and each $H_i \subseteq A_k$ for some k, $[a, b] = \bigcup_{k=1}^{n} A_k = \bigcup_{i=1}^{\infty} H_i$. We will now show that each of the H_i's are pairwise disjoint. We see that by DeMorgan's Laws,

$$
\begin{aligned}
H_1 \cap H_2 &= E_1 \cap (E_2 - E_1) = \emptyset \\
H_2 \cap H_3 &= (E_2 - E_1) \cap [E_3 - (E_1 \cup E_2)] \\
&= (E_2 - E_1) \cap [(E_3 - E_1) \cap (E_3 - E_2)] = \emptyset \\
&\vdots \\
H_i \cap H_{i-1} &= [E_i - (E_1 \cup \cdots \cup E_{i-1})] \cap [E_{i-1} - (E_1 \cup \cdots \cup E_{i-2})] \\
&= (E_i - E_1) \cap [E_i - (E_2 \cup \cdots \cup E_{i-1})] \\
&\cap (E_{i-1} - E_1) \cap [E_{i-1} - (E_2 \cup \cdots \cup E_{i-2})] = \emptyset.
\end{aligned}
$$

Hence, the H_i's are pairwise disjoint.

Now, define

$$\Pi_1 = \{i : H_i \subseteq A_1\}$$

and

$$\Pi_k = \{i : \mathbb{Z}^+ - (\Pi_1 \cup \cdots \cup \Pi_{k-1}) : H_i \subseteq A_k\},$$

for $2 \leq k \leq n$. Also, let

$$B_k = \bigcup_{i \in \Pi_k} H_i.$$

We will show that the B_k's are pairwise disjoint F_σ sets, $B_k \subseteq A_k$ for each k, and conclude that $[a, b] = \bigcup_{k=1}^{n} B_k$.

Since each B_k is made up of pairwise disjoint F_σ sets H_i, it follows that the B_k's are pairwise disjoint F_σ sets as well. Also, since each $H_i \subseteq A_k$ for some k, it follows that $B_k \subseteq A_k$ for all k. Lastly, since

$[a, b] = \bigcup_{i=1}^{\infty} H_i$, we ask if all of the H_i's are used to build each of the B_k's. The answer is yes, since by definition of B_k, for a given j, H_j is used to build B_l where $l = \min\{k : H_j \subseteq A_k\}$. Hence, $[a, b] = \bigcup_{k=1}^{n} B_k$. This completes the proof. $\square$

We digress briefly to present a theorem concerning continuous functions. This will "motivate" an analogous result for Baire one functions. We first present a definition and several results that will be used in the proof of Theorem 3.

Definition 5. [4] *Let A be a set of real numbers.*

(1) *A set $E \subseteq A$ is* ***open in*** *A if there is an open set O such that $E = A \cap O$.*
(2) *A set $E \subseteq A$ is* ***closed in*** *A if there is a closed set F such that $E = A \cap F$.*

Lemma 6. [4] *Let A be a set of real numbers.*

(1) *A set $E \subseteq A$ is open in A if and only if $A - E$ is closed in A.*
(2) *A set $E \subseteq A$ is closed in A if and only if $A - E$ is open in A.*
(3) *If A is an open set, then a set $E \subseteq A$ is open in A if and only if it is open.*
(4) *If A is an closed set, then a set $E \subseteq A$ is closed in A if and only if it is closed.*

Lemma 7. [4] *Let E be a set of real numbers. A point x is a limit point of E if and only if there exists a sequence of points in $E - \{x\}$ that converges to x.*

We are now ready to prove the following result.

Theorem 3. *A function $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ if and only if the sets $\{x \in [a, b] : f(x) < r\}$ and $\{x \in [a, b] : f(x) > r\}$ are open in $[a, b]$ for each real number r.*

Proof. Suppose that the sets $\{x \in [a, b] : f(x) < r\}$ and $\{x \in [a, b] : f(x) > r\}$ are open in $[a, b]$ for each real number r. Let $c \in [a, b]$ and let $\epsilon > 0$. By definition, we see that there exist open sets O_1 and O_2 such that

$$\{x \in [a, b] : f(x) < f(c) + \epsilon\} = [a, b] \cap O_1$$
$$\{x \in [a, b] : f(x) > f(c) - \epsilon\} = [a, b] \cap O_2$$

where $r = f(c) \pm \epsilon$.

Since $c \in [a, b]$, and O_1 and O_2 are open sets, the set $O = O_1 \cap O_2$ is an open set that contains the point c. Next, choose $\delta > 0$ such that the interval $(c-\delta, c+\delta) \subseteq O$. If $|x-c| < \delta$ and $x \in [a,b]$, then $x \in [a,b] \cap O_1$ and $x \in [a,b] \cap O_2$ by the definition of O. Hence,

$$f(c) - \epsilon < f(x) < f(c) + \epsilon.$$

It follows that $|f(x) - f(c)| < \epsilon$ for all $x \in [a,b]$ such that $|x-c| < \delta$. This shows that f is continuous on $[a,b]$.

Next, suppose that f is continuous on $[a,b]$ and let r be a real number. To show that the sets $\{x \in [a,b] : f(x) < r\}$ and $\{x \in [a,b] : f(x) > r\}$ are open in $[a,b]$ for each real number r, we will use properties (2) and (4) of Lemma 6, and Lemma 7 to show that the sets

$$\{x \in [a,b] : f(x) \leq r\}$$
$$\{x \in [a,b] : f(x) \geq r\}$$

are closed in $[a,b]$, that is each set contains all of its limit points.

First, let $E = \{x \in [a,b] : f(x) \geq r\}$ and suppose w is a limit point of E, that is, for each positive number m, the set $(w-m, w+m) \cap E$ contains a point of E other than w. By Lemma 7, there exists a sequence $\{x_n\}$ in $E - \{w\}$ that converges to w. By assumption, f is continuous at w, so the sequence $\{f(x_n)\}$ converges to $f(w)$. Since $f(x_n) \geq r$ for all n, we have $f(w) \geq r$. Thus, w is in E.

Likewise, let $G = \{x \in [a,b] : f(x) \leq r\}$ and suppose d is a limit point of G. By Lemma 7, there exists a sequence $\{y_n\}$ in $G - \{d\}$ that converges to d. By assumption f is continuous at d, so the sequence $\{f(y_n)\}$ converges to $f(d)$. Since $f(y_n) \leq r$ for all n, we have $f(d) \leq r$. Thus, d is in G.

We have thus shown that E and G each contain all of their limit points. By definition the sets E and G are closed. □

We now introduce the concept of the characteristic function of a set $B \subseteq \mathbb{R}$. Lemma 8 is called the *Tietze Extension Theorem*, which is a result from analysis (or, more generally, topology) [4]. Lemma 9 is a result that uses both the definition of the characteristic function of a set and the Tietze Extension Theorem, and will also be used in the final theorem of this section.

Definition 6. *Let $B \subseteq \mathbb{R}$. The **characteristic function** of B is the real-valued function $\chi_B : \mathbb{R} \to \mathbb{R}$ defined by the equation*

$$\chi_B(x) = \begin{cases} 1, & \text{if } x \in B; \\ 0, & \text{if } x \in \mathbb{R} - B. \end{cases}$$

Lemma 8 (Tietze Extension Theorem)**.** *Let E be a closed set. If a function $f : E \to \mathbb{R}$ is continuous on E, then there exists a continuous function $f_e : \mathbb{R} \to \mathbb{R}$ such that $f(x) = f_e(x)$ for all $x \in E$.*

Lemma 9. *Suppose that $[a, b] = \bigcup_{k=1}^{n} B_k$ where each B_k is an F_σ set and the B_k's are pairwise disjoint. Let*

$$f(x) = \sum_{k=1}^{n} c_k \chi_{B_k}(x)$$

for $x \in [a, b]$. Then f is a Baire one function.

Proof. For each k let $B_k = \bigcup_{i=1}^{\infty} E_i^k$ where the sequence $\{E_i^k\}$ is a nondecreasing sequence of closed sets. This is possible because by assumption, each B_k is an F_σ set. For each positive integer m, let

$$f_m = \sum_{k=1}^{n} c_k \chi_{E_m^k}.$$

We will verify that the restriction of f_m to the closed set $\bigcup_{k=1}^{n} E_m^k$ is continuous.

Suppose there exists a sequence $\{x_l\}$ that converges to $x_0 \in \bigcup_{k=1}^{n} E_m^k$. Let $x_0 \in E_m^j$ for some fixed j, $1 \leq j \leq n$. We claim that all but finitely many of the x_l's are in E_m^j. Suppose not, then there exists a subsequence $\{x_{l_n}\} \subseteq E_m^k$, such that $\{x_{l_n}\}$ converges to $x_0 \in E_m^k$. Since each E_m^k is closed and disjoint, the set E_m^j contains finitely many points x_l.

Thus, for sufficiently large l, $f(x_l) = c_j = f(x_0)$. This certainly implies that $f(x_l)$ converges to $f(x_0)$. By definition, f is continuous at x_0 and hence on all of $\bigcup_{k=1}^{n} E_m^k$.

Now let g_m be the linear extension of f_m from $\bigcup_{k=1}^{n} E_m^k$ to all of $\mathbb{R}$. By the Tietze Extension Theorem g_m is continuous on $\mathbb{R}$. Hence, g_m is continuous on $[a, b]$.

Finally, since

$$f(x) = \lim_{m \to \infty} g_m(x)$$

for all $x \in [a, b]$, the sequence $\{g_m\}$ converges pointwise to f on $[a, b]$. It follows that f is the pointwise limit of a sequence of continuous functions. By definition, f is Baire one. □

We now present the main result of this section, and as stated earlier, it "looks" very similar to Theorem 3 concerning continuous functions. It follows from the next theorem that a function f is Baire one if and only if two sets involving the inverse image of f are F_σ sets.

Theorem 4. *A function $f : [a, b] \to \mathbb{R}$ is a Baire one function if and only if the sets $\{x \in [a, b] : f(x) < r\}$ and $\{x \in [a, b] : f(x) > r\}$ are F_σ sets for each real number r.*

Proof. Suppose that f is a Baire one function. Let $\{f_n\}$ be a sequence of continuous functions that converges pointwise to f on $[a, b]$, and let r be a real number. It can be shown that

$$\{x \in [a, b] : f(x) < r\} = \bigcup_{k=1}^{\infty} \bigcup_{p=1}^{\infty} \left[\bigcap_{n=p}^{\infty} \left\{ x \in [a, b] : f_n(x) \leq r - \frac{1}{k} \right\} \right].$$

Since f_n is a sequence of continuous functions, the set

$$\left\{ x \in [a, b] : f_n(x) \leq r - \frac{1}{k} \right\}$$

is closed, since the inverse image of a closed set is closed [4]. It follows that the set $\{x \in [a, b] : f(x) < r\}$ is an F_σ set since the countable union of intersections of closed sets is an F_σ set. Since f is Baire one by assumption, $-f$ is Baire one by properties of Baire one functions. Hence, the set $\{x \in [a, b] : f(x) > r\}$ is also an F_σ set using a similar argument.

For the converse, suppose that f is a bounded function such that the sets $\{x \in [a, b] : f(x) < r\}$ and $\{x \in [a, b] : f(x) > r\}$ are F_σ sets for each real number r. We will show the general case (when f is unbounded) later on. We use the following construction to produce a sequence $\{f_n\}$ of Baire one functions. Choose $M > 0$ such that $|f(x)| < M$ for all $x \in [a, b]$. Next, fix n and let

$$y_k = -M + \frac{2Mk}{n},$$

for $0 \leq k \leq n$. Also, let

$$A_k = \{x \in [a, b] : y_{k-1} < f(x) < y_{k+1}\},$$

for $1 \leq k \leq n$. Let

$$\begin{aligned} A &= \{x \in [a, b] : f(x) < y_{k+1}\} \\ B &= \{x \in [a, b] : f(x) \leq y_{k-1}\}. \end{aligned}$$

We will show that $A_k = A \cap B^C$ is an F_σ set, where B^C denotes the complement of B. By hypothesis, A is an F_σ set. We see that $B^C = \{x \in [a, b] : f(x) > y_{k-1}\}$ is an F_σ set by the same argument.

Since the intersection of two F_σ sets is also an F_σ set, it follows that A_k is an F_σ set as well. Also by hypothesis,

$$[a,b] = \bigcup_{k=1}^{n-1} A_k.$$

By Lemma 5, $[a,b] = \bigcup_{k=1}^{n-1} B_k$, where each B_k is an F_σ set, $B_k \subseteq A_k$ for each k, and the B_k's are pairwise disjoint. For each n, define $f_n : [a,b] \to \mathbb{R}$ as

$$f_n(x) = \sum_{k=1}^{n-1} y_k \chi_{B_k}(x).$$

By Lemma 9, f_n is a Baire one function.

The sets $B_1, B_2, ..., B_{n-1}$ are pairwise disjoint and partition $[a,b]$. Next, fix n and $x \in [a,b]$. There exists j, $1 \le j \le n$, such that $x \in B_j$ and $B_j \subseteq A_j$. By definition, $f_n(x) = y_j$, for $x \in B_j$. We know that

$$y_{j-1} < f(x) < y_{j+1}, \qquad y_{j-1} < y_j < y_{j+1}.$$

Since $f_n(x) = y_j$, we have,

$$y_{j-1} < f_n(x) < y_{j+1}.$$

Thus, by definition of y_k above,

$$\begin{aligned} |f_n(x) - f(x)| &< \frac{1}{2}|y_{j+1} - y_{j-1}| \\ &= \frac{2M}{n}, \end{aligned}$$

for all $x \in [a,b]$ and all $n \in \mathbb{N}$. By definition, the sequence $\{f_n\}$ converges uniformly to f on $[a,b]$. By Theorem 2, f is a Baire one function.

For the general case (where f is not necessarily bounded), let $h : \mathbb{R} \to (0,1)$ be a continuous increasing function. For each real number r we have,

$$(h \circ f)^{-1}(r,\infty) = f^{-1}(h^{-1}(r,\infty))$$

by properties of functions. We will show that,

$$(h \circ f)^{-1}(r,\infty) = \begin{cases} [a,b], & \text{if } r \le 0; \\ f^{-1}(h^{-1}(r),\infty), & \text{if } 0 < r < 1; \\ \emptyset, & \text{if } r \ge 1, \end{cases}$$

where $f : [a,b] \to \mathbb{R}$, $h : \mathbb{R} \to (0,1)$, and $(h \circ f) : [a,b] \to (0,1)$. We note that $h \circ f$ is a bounded function.

For $r \leq 0$, we have,

$$\begin{aligned} h^{-1}((r,\infty)) &= \{x \in \mathbb{R} : h(x) \in (r,\infty)\} \\ &= \{x \in \mathbb{R} : h(x) > r\} = \mathbb{R}. \end{aligned}$$

Thus, $(h \circ f)^{-1}(r,\infty) = f^{-1}(h^{-1}(r,\infty)) = f^{-1}(\mathbb{R}) = [a,b]$.

For $0 < r < 1$, we see that,

$$\begin{aligned} h^{-1}((r,\infty)) &= \{x \in \mathbb{R} : h(x) \in (r,\infty)\} \\ &= \{x \in \mathbb{R} : h(x) > r\} \\ &= \{x \in \mathbb{R} : x > h^{-1}(r)\} \\ &= (h^{-1}(r),\infty). \end{aligned}$$

Thus, $(h \circ f)^{-1}(r,\infty) = f^{-1}(h^{-1}(r,\infty)) = f^{-1}(h^{-1}(r),\infty)$.

Finally, for $r \geq 1$,

$$\begin{aligned} h^{-1}((r,\infty)) &= \{x \in \mathbb{R} : h(x) \in (r,\infty)\} \\ &= \{x \in \mathbb{R} : h(x) > r\} = \emptyset \end{aligned}$$

Thus, $(h \circ f)^{-1}(r,\infty) = f^{-1}(h^{-1}(r,\infty)) = f^{-1}(\emptyset) = \emptyset$.

Since,

$$\begin{aligned} (h \circ f)^{-1}(r,\infty) &= f^{-1}(h^{-1}(r,\infty)) \\ &= \{x \in [a,b] : f(x) \in (h^{-1}(r),\infty)\} \\ &= \{x \in [a,b] : f(x) > h^{-1}(r)\}, \end{aligned}$$

it follows that $h \circ f$ satisfies the condition in the theorem and hence is a Baire one function. By properties of functions,

$$f = h^{-1} \circ (h \circ f).$$

Since h is continuous and increasing, it is Baire one. By properties of limits, the composition of a continuous and a Baire one function is also Baire one. It follows that f is a Baire one function. This completes the proof. □

As an illustration of Theorem 4, consider the characteristic function χ_Q on the set $\mathbb{Q}$ of rational numbers,

$$\chi_Q(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{if } x \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

We know that the set of irrational numbers $\mathbb{R}-\mathbb{Q}$ is not an F_σ set [4]. Let $r=1$. The set $\{x \in [a,b] : \chi_Q < 1\} = \{x \in [a,b] : \chi_Q = 0\} = \mathbb{R}-\mathbb{Q}$ is not an F_σ set. Hence, the function χ_Q is not a Baire one function.[1]

6. Points of Continuity of f

Recall that many Baire one functions are discontinuous, but certainly not all discontinuous functions are necessarily Baire one. How discontinuous are Baire one functions? Since the nowhere continuous function χ_Q is not a Baire one function, we may wonder whether Baire one functions must be continuous somewhere over its domain. In fact, we will show that if f is a Baire one function, then every closed interval contains a point of continuity of f.

We need some new definitions to help us with our proof. In addition, the following lemmas will be useful. Let $f : \mathbb{R} \to \mathbb{R}$ be a fixed function.

Definition 7. *Let $A \subseteq \mathbb{R}$. The extended (possibly infinite) real number $\omega(A) = \sup\{|f(x) - f(y)| : x, y \in A\}$ is called the **oscillation** of f on A.*

One can think of the oscillation of a function f on a set A as roughly the largest distance between the maximum and minimum values of $f(x)$ for all $x \in A$. We can also define the oscillation of a function f at a point. This is shown as follows.

Definition 8. *For $x_0 \in \mathbb{R}$, let $A_\delta = (x_0 - \delta, x_0 + \delta)$ for each $\delta > 0$. The **oscillation of** f **at** x_0 is the number $\omega(x_0) = \lim_{\delta \to 0} \omega(A_\delta)$.*

Let's use this definition to show a simple (yet important) result involving continuity.

Lemma 10. *f is continuous at x_0 if and only if $\omega(x_0) = 0$.*

Proof. First, suppose f is continuous at x_0 and let $\epsilon > 0$. By definition, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \frac{\epsilon}{3}$. Then, for all $x, y \in A_\delta$

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(x_0)| + |f(x_0) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3} < \epsilon. \end{aligned}$$

It follows that $\sup\{|f(x) - f(y)| : x, y \in A_\delta\} \leq \frac{2}{3}\epsilon < \epsilon$. Since $\epsilon > 0$ was arbitrary, we have $\omega(x_0) = 0$.

For the converse, suppose $\omega(x_0) = 0$ and let $\epsilon > 0$. There exists $\delta_0 > 0$ such that $|\omega(A_\delta)| < \epsilon$ for all x such that $|x - x_0| < \delta_0$. Thus

[1] While not Baire one, it can be shown that χ_Q is Baire class two. See [7].

by definition, $\lim_{\delta \to 0} \sup\{|f(x) - f(y)| : x, y \in A_\delta\} = 0$ and $|f(x) - f(x_0)| < \epsilon$. Therefore, x is continuous at x_0. □

The next lemma will be used in our proof of Theorem 5. It is mentioned in discussion notes [5], and it won't be proved here. It is a version of a famous theorem called the Baire Category Theorem.

Lemma 11. *Suppose that $\{D_n\}$ is a sequence of closed subsets of $\mathbb{R}$ and that $[a, b] = \bigcup_{n=1}^{\infty} D_n$. Then at least one of the sets D_n contains a closed interval.*

We are now ready to prove the following.

Theorem 5. *If $f : \mathbb{R} \to \mathbb{R}$ is a Baire one function, then every closed interval contains a point of continuity of f.*

Proof. Let I_0 be a closed interval and $\{f_n\}$ be a sequence of continuous functions on $\mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$. We will prove that I_0 contains a point of continuity of f.

Let $\epsilon > 0$. We first show that there exists a closed interval $I_1 \subseteq I_0^\circ$ such that $\omega(I_1) \leq \epsilon$, where I_0° is the interior of I_0 (or the non-endpoints of I_0). For $m, n \in \mathbb{N}$, let

$$A_{nm} = \left\{ x \in I_0 : |f_n(x) - f_{n+m}(x)| \leq \frac{\epsilon}{3} \right\}. \tag{1}$$

Since f_n is a sequence of continuous functions and the inverse image of a closed set is closed, the set A_{nm} is closed [4]. By properties of closed sets, the set

$$D_n = \bigcap_{m=1}^{\infty} A_{nm}$$

is closed for all n We will now show that

$$I_0 = \bigcup_{n=1}^{\infty} D_n.$$

We note that $D_n \subseteq I_0$ by definition. Hence, we need to show that $I_0 \subseteq D_n$ for some n. In other words, given a fixed $x \in I_0$, we must find an N such that $x \in A_{Nm}$ for all m.

Let $\epsilon > 0$. Since by hypothesis, $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$, there exists a positive integer N such that $|f_n(x) - f(x)| \leq \frac{\epsilon}{6}$ for all $n \geq N$. It follows that,

$$\begin{aligned}|f_N(x) - f_{N+m}(x)| &\leq |f_N(x) - f(x)| + |f(x) - f_{N+m}(x)| \\ &\leq \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3}.\end{aligned}$$

Thus, $x \in A_{Nm}$ for all m, $x \in D_N$, and $I_0 \subseteq D_n$ for some n. By Lemma 11, there exists $n \in \mathbb{N}$ and a closed interval J such that $J \subseteq D_n$.

Next, we note that for a fixed n, $f_{n+m}(x)$ converges pointwise to $f(x)$ for all $x \in I_0$. It follows from (1) that

$$|f_n(x) - f(x)| \leq \frac{\epsilon}{3} \qquad \text{for all } x \in J. \tag{2}$$

We now show that there exists a closed interval $I_1 \subseteq J^\circ \subseteq I_0^\circ$ such that $\omega(I_1, f) \leq \epsilon$. First, we show that $\omega(I_1, f_n) \leq \frac{\epsilon}{3}$. This new notation denotes the oscillation of f_n over the interval I_1. Since each f_n is continuous on J, then f_n is uniformly continuous on J [4]. Let $\epsilon > 0$. There exists $\delta > 0$ such that $|f_n(x) - f_n(y)| \leq \frac{\epsilon}{3}$ for all $x, y \in J$ that satisfies $|x - y| \leq \delta$. Since $I_1 \subseteq J$ we have,

$$\omega(I_1, f_n) = \sup\{|f_n(x) - f_n(y)| : x, y \in I_1\} \leq \frac{\epsilon}{3}. \tag{3}$$

Now let $x, y \in I_1$. Using equations (2) and (3) we have,

$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon.\end{aligned}$$

We see that,

$$\omega(I_1, f) = \sup\{|f(x) - f(y)| : x, y \in I_1\} \leq \epsilon.$$

Using the construction above, we construct a sequence $\{I_k\}$ of closed intervals such that $I_{k+1} \subseteq I_k^\circ \subseteq I_0^\circ$ for $k = 1, 2, ...$ and

$$\omega(I_k, f) \leq \frac{1}{k}. \tag{4}$$

By the Nested Intervals Theorem [4], the set

$$\bigcap_{k=1}^{\infty} I_k$$

contains at least one point x_0 and $x_0 \in I_0$. It follows from (4) that $\omega(x_0) = 0$. By Lemma 10, f is continuous at x_0. This completes the proof. $\square$

7. Baire One and First-Return Recoverability

A well known fact from analysis is that a continuous function $f : [0,1] \to [0,1]$ is uniquely determined by its values on any countable dense set in $[0,1]$ (we define what it means for a set to be dense later on) [9]. We extend this idea to Baire one functions and ask whether they too can be determined by its values on any countable set. Since not all Baire one functions are continuous, the answer is no, but there are certain countable dense sets that give us some nice results. We conclude with a characterization of Baire one functions using the notion of *first return recoverability*. First, let's state several definitions.

Definition 9 (Dense set)**.** [8] *A set $D \subseteq X$ is **dense** in X if for any point $x \in X$, any neighborhood of x contains at least one point from D.*

Equivalently, D is dense in X if and only if $\overline{D} = X$, where $\overline{D}$ is the closure of D.

Underlying our characterization is the idea of what we call a *trajectory*. This is defined below.

Definition 10. *A **trajectory** is any sequence $\{x_n\}$ of distinct points in $[0,1]$, whose range is dense in $[0,1]$.*

Any countable dense set $J \subseteq [0,1]$ is called a *support set* and any enumeration of D is also a trajectory. If the value of a continuous function $f : [0,1] \to \mathbb{R}$ at each point in a trajectory is known, then there is an algorithm that computes the value of the function at *any* point in $[0,1]$.

Definition 11. *Let $\{x_n\}$ be a fixed trajectory and let $y \in [0,1]$. Suppose $\rho > 0$ and let $B_\rho(y) = \{x \in [0,1] : |x - y| < \rho\}$. Also, let $r(B_\rho(y))$ be the first element of the trajectory in $B_\rho(y)$. The **first return route to** y, denoted $\mathcal{R}_y = \{y_k\}$, is given by the recursive formula*

$$y_1 = x_0,$$

$$y_{k+1} = \begin{cases} r(B_{|y-y_k|}(y)), & \text{if } y \neq y_k \\ y_k, & \text{if } y = y_k. \end{cases}$$

The function $f : [0,1] \to \mathbb{R}$ is called *first return recoverable with respect to* $\{x_n\}$ if for each $y \in [0,1]$,

$$\lim_{k\to\infty} f(y_k) = f(y),$$

and f is *first return recoverable* if there exists a trajectory $\{x_n\}$ such that f is first return recoverable with respect to $\{x_n\}$.

It can be shown that a function $f : [0,1] \to \mathbb{R}$ is continuous if and only if it is first-return recoverable with respect to *every* trajectory. However, if we weaken this condition, we obtain the following theorem characterizing Baire class one functions. Before we present the theorem, we introduce the definition of a G_δ set. It will be used in the proof of our last theorem.

Definition 12 (G_δ set)**.** [4] *A subset $G \subseteq \mathbb{R}$ is called a* $\mathbf{G}_\delta$ ***set*** *if G is a countable intersection of open sets. In other words, $G = \bigcap_{n=1}^{\infty} G_n$ where each G_n is an open subset of $\mathbb{R}$.*

Note that by DeMorgan's Laws, the complement of a G_δ set is F_σ [4].

Theorem 6. *A function $f : [0,1] \to \mathbb{R}$ is first return recoverable if and only if it is of Baire class one.*

Proof. We will prove the forward direction. The other direction is given in [6], which we leave to the reader. Suppose $f : [0,1] \to \mathbb{R}$ is first return recoverable with respect to a trajectory $\{x_n\}$. It suffices to show that for each real number r, the set $A = \{x \in [0,1] : f(x) \geq r\}$ is a G_δ set. If we can prove this, then we know that $-f$ is also first return recoverable and that $\{x \in [0,1] : -f(x) \geq -r\}$ is a G_δ set, or equivalently $\{x \in [0,1] : f(x) \leq r\}$ is a G_δ set. It follows from Theorem 4 that f is Baire one.

Since f is first return recoverable with respect to a trajectory $\{x_n\}$, $f(x) = \lim_{k\to\infty} f(y_k)$ for all $x \in [0,1]$ where $\{y_k\} = \mathcal{R}_y$, as defined in the definition above. Fix $r \in \mathbb{R}$. We assume the following.

(1) For each positive integer k, let $I_k = (r - \frac{1}{k}, \infty)$.
(2) Let $\{x_{n_j}\}$ be a subsequence of the trajectory $\{x_n\}$ consisting of those $x_n \in f^{-1}(I_k)$. In other words, those x_n that satisfy $f(x_n) > r - \frac{1}{k}$.
(3) For each $j \in \mathbb{N}$, let $d_j = \min\{\frac{1}{2}|x_{n_j} - x_i| : i < n_j\}$, and let $O_j = (x_{n_j} - d_j, x_{n_j} + d_j)$.

Let

$$H_k = \bigcap_{i=1}^{\infty} \left[\bigcup_{j=i}^{\infty} O_j \cup \{x_{n_1}, x_{n_2}, \ldots, x_{n_{i-1}}\} \right].$$

From (1), (2), and (3) above, we have

$$A = \bigcap_{k=1}^{\infty} H_k.$$

We now show that for each fixed k, H_k is G_δ. First, since the union of any collection of open sets is open, $\bigcup_{j=i}^{\infty} O_j$ is an open set. Second, the set $\{x_{n_1}, x_{n_2}, \ldots, x_{n_{i-1}}\}$ is G_δ because it has a finite number of elements [4]. Hence, it is equal to a countable intersection of open sets. Let $G = \bigcup_{j=i}^{\infty} O_j$ and the set $\{x_{n_1}, x_{n_2}, \ldots, x_{n_{i-1}}\} = \bigcap_{n=1}^{\infty} G_n$. It follows that,

$$
\begin{aligned}
H_k &= G \cup \left[\bigcap_{n=1}^{\infty} G_n \right] \\
&= \bigcap_{n=1}^{\infty} \left[G \cup G_n \right]
\end{aligned}
$$

where G and each G_n are open sets. Thus, H_k is a countable intersection of G_δ sets and is a G_δ set. It follows easily that A is a G_δ set as well. $\square$

8. Conclusion

While the definition of a Baire one function is fairly straightforward, this paper shows that some of its properties and examples certainly are not. In fact, most research papers dealing with this topic require graduate-level analysis and/or topology in order to fully appreciate (and understand) the results. Hence, many of the (perhaps) more interesting results are not presented here.

With that said, there is great potential for further study of Baire functions once one becomes more mathematically mature. Therefore, this paper can be regarded as a starting point to a future, more in-depth project or thesis.

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