PROVING COMPLETENESS OF THE HAUSDORFF INDUCED METRIC SPACE

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Abstract

Given a metric space (X, d), we may define a new metric space with Hausdorff metric h on the set $\mathcal{K}$ of the collection of all nonempty compact subsets of X. We show that if (X, d) is complete, then the Hausdorff metric space $(\mathcal{K}, h)$ is also complete.

Introduction

The Hausdorff distance, named after Felix Hausdorff, measures the distance between subsets of a metric space. Informally, the Hausdorff distance gives the largest length out of the set of all distances between each point of a set to the closest point of a second set. Given any metric space, we find that the Hausdorff distance defines a metric on the space of all nonempty compact subsets of the metric space. We find that there are many interesting properties of this metric space, which will be our focus in this paper. The first property is that the Hausdorff induced metric space is complete if our original metric space is complete. Similarly, the second property we explore is that if our original metric space is compact, then our Hausdorff induced metric space is also compact.

In the next section, we provide some definitions and theorems necessary for understanding this paper. We then define the Hausdorff distance in the following section, and examine its properties through some examples and short proofs. We find that the Hausdorff distance satisfies the conditions for a metric on a space of nonempty compact subsets of a metric space. Finally, in our last section, we prove that if our original metric space is complete then the Hausdorff induced metric space is also complete. We further show that $(\mathcal{K}, h)$ is compact when (X, d) is compact.

Preliminaries

The concepts in this paper should be familiar to anyone who has taken a course in Real Analysis. The notation and terminology in this paper will come from Gordon's *Real Analysis: A First Course* [1]. Therefore, we expect the reader to be familiar with the following concepts regarding metric spaces and real numbers.

Definition 2.1 Let S be a nonempty set of real numbers that is bounded below. The number α is the *infimum* of S if α is a lower bound of S and any number greater than α is not a lower bound of S. We will write $\alpha = \inf S$. The definition of the *supremum* of S is analogous and will be denoted by $\sup S$.

Completeness Axiom Each nonempty set of real numbers that is bounded below has an infimum. Similarly, any nonempty set of real numbers that is bounded above has a supremum.

The reader may be more familiar with the following definitions when applied to the metric space $(\mathbb{R}, d)$, where $d(x, y) = |x - y|$. However, with the exclusion of some examples, for the majority of this paper we will be working in a general metric space. Thus our definitions will be given with respect to any metric space (X, d).

Definition 2.2 A **metric space** (X, d) consists of a set X and a function $d : X \times X \to \mathbb{R}$ that satisfies the following four properties.

(1) $d(x, y) \geq 0$ for all $x, y \in X$.

(2) $d(x, y) = 0$ if and only if $x = y$.

(3) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The function d, which gives the distance between two points in X, is called a **metric**. For example, a metric on the set of real numbers is $d(x, y) = |x - y|$. It is easily verified that d satisfies the four properties listed above.

For the next set of definitions, let (X, d) be a metric space.

Definition 2.3 Let $v \in X$ and let $r > 0$. The **open ball** centered at v with radius r is defined by $B_d(v, r) = \{x \in X : d(x, v) < r\}$.

Definition 2.4 A set $E \subseteq X$ is **bounded** in (X, d) if there exist $x \in X$ and $M > 0$ such that $E \subseteq B_d(x, M)$.

Definition 2.5 A set $K \subseteq X$ is **totally bounded** if for each $\epsilon > 0$ there is a finite subset $\{x_i : 1 \leq i \leq n\}$ of K such that $K \subseteq \bigcup_{i=1}^{n} B_d(x_i, \epsilon)$.

For the following definitions, let $\{x_n\}$ be a sequence in a metric space (X, d).

Definition 2.6 The sequence $\{x_n\}$ **converges** to $x \in X$ if for each $\epsilon > 0$ there exists a positive integer N such that $d(x_n, x) < \epsilon$ for all $n \geq N$. We say $\{x_n\}$ converges if there exists a point $x \in X$ such that $\{x_n\}$ converges to x.

Definition 2.7 The sequence $\{x_n\}$ is a **Cauchy sequence** if for each $\epsilon > 0$ there exists a positive integer N such that $d(x_n, x_m) < \epsilon$ for all $m, n \geq N$.

It is easy to verify that every convergent sequence is a Cauchy sequence.

Definition 2.8 A metric space (X, d) is **complete** if every Cauchy sequence in (X, d) converges to a point in X.

An example of a metric space that is not complete is the space $(\mathbb{Q}, d)$ of rational numbers with the standard metric given by $d(x, y) = |x - y|$. However, the space

$\mathbb{R}$ of real numbers and the space $\mathbb{C}$ of complex numbers under the same metric $d(x, y) = |x - y|$ are complete.

Definition 2.9 A set $K \subseteq X$ is **sequentially compact** in (X, d) if each sequence in K has a subsequence that converges to a point in K.

Note that by Theorem 8.59 in [1], a subset of a metric spaces is compact if and only if it is sequentially compact; therefore, we will use the concepts of sequentially compact and compact interchangeably throughout this paper.

Definition 2.10 The point x is a **limit point** of a set E if for each $r > 0$, the set $E \cap B_d(x, r)$ contains a point of E other than x.

As an alternative to the definition, Theorem 8.49 in [1] states that x is limit point of the set E if and only if there exists a sequence of points in $E \backslash \{x\}$ that converges to x. This theorem provides us with the opportunity to choose a sequence converging to x, which will be useful in proving that a set is closed.

Definition 2.11 A set E is **closed** in (X, d) if E contains all of its limit points.

Definition 2.12 The **closure** of E, denoted $\overline{E}$, is the set $E \cup E'$, where E' is the set of all limit points of E.

The following two results and lemma are placed in this section to be referred to in later proofs. In addition they will serve as an introduction to proofs that use the definition of convergent sequences and the triangle inequality.

Result 1: Let $\{x_n\}$ and $\{y_n\}$ be sequences in a metric space (X, d). If $\{x_n\}$ converges to x and $\{y_n\}$ converges to y, then $\{d(x_n, y_n)\}$ converges to $d(x, y)$.

Proof. Let $\epsilon > 0$. Since $\{x_n\}$ converges to x, by definition there exists a positive integer N_1 such that $d(x_n, x) < \frac{\epsilon}{2}$ for all $n \geq N_1$. Similarly, since $\{y_n\}$ converges to y, there exists a positive integer N_2 such that $d(y_n, y) < \frac{\epsilon}{2}$ for all $n \geq N_2$. Choose $N = \max\{N_1, N_2\}$. Then for all $n \geq N$, we find that

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n) < \frac{\epsilon}{2} + d(x, y) + \frac{\epsilon}{2} = d(x, y) + \epsilon,$$

and

$$d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y) < \frac{\epsilon}{2} + d(x_n, y_n) + \frac{\epsilon}{2} = d(x_n, y_n) + \epsilon.$$

Together these inequalities imply $|d(x_n, y_n) - d(x, y)| < \epsilon$ for all $n \geq N$. Therefore, $\{d(x_n, y_n)\}$ converges to $d(x, y)$. □

Result 2: If $\{z_k\}$ is a sequence in a metric space (X, d) with the property that $d(z_k, z_{k+1}) < 1/2^k$ for all k, then $\{z_k\}$ is a Cauchy sequence.

Proof. Let $\epsilon > 0$ and choose a positive integer N such that $\frac{1}{2^{N-1}} < \epsilon$. Then for all $n > m \geq N$ we find that

$$\begin{aligned} d(z_m, z_n) &\leq d(z_m, z_{m+1}) + d(z_{m+1}, z_{m+2}) + \cdots + d(z_{n-1}, z_n) \\ &< \frac{1}{2^m} + \frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{n-1}} \\ &< \sum_{k=m}^{\infty} \frac{1}{2^k} = \frac{1}{2^{m-1}} \leq \frac{1}{2^{N-1}} < \epsilon. \end{aligned}$$

It follows that $\{z_k\}$ is a Cauchy sequence. □

Lemma 1: Let (X, d) be a metric space and let A be a closed subset of X. If $\{a_n\}$ converges to x and $a_n \in A$ for all n, then $x \in A$.

Proof. Suppose $\{a_n\}$ is a sequence that converges to x and $a_n \in A$ for all n. There are two cases to consider. If there exists a positive integer n such that $a_n = x$, then it is clear $x \in A$. If there does not exist a positive integer n such that $a_n = x$, then x is a limit point of A by Theorem 8.49 in [1]. Since A is closed, $x \in A$. □

Construction of the Hausdorff Metric

We now define the Hausdorff metric on the set of all nonempty, compact subsets of a metric space. Let (X, d) be a complete metric space and let $\mathcal{K}$ be the collection of all nonempty compact subsets of X. Note that $\mathcal{K}$ is closed under finite unions and nonempty intersections. For $x \in X$ and $A, B \in \mathcal{K}$, define

$$r(x, B) = \inf\{d(x, b) : b \in B\} \text{ and } \rho(A, B) = \sup\{r(a, B) : a \in A\}.$$

Note that r is nonnegative and exists by the Completeness Axiom since $d(a, b) \geq 0$ by the definition of a metric space. Since r exists and is nonnegative, then both $\rho(A, B)$ and $\rho(B, A)$ exist and are nonnegative. In addition, we define the *Hausdorff distance* between sets A and B in $\mathcal{K}$ as

$$h(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

Before proving that h defines a metric on the set $\mathcal{K}$, let us consider a few examples to get a grasp on how these distances work. Consider the following example of closed interval sets in $(\mathbb{R}, d)$, where $d(x, y) = |x - y|$.

Example Let $A = [0, 20]$ and let $B = [22, 31]$.

We find that $r(x, B)$ is going to be the infimum of the set of distances from each $a \in A$ to the closest point in B. As an example of one of these distances, consider $a = 12$. Then $r(12, B) = \inf\{d(12, b) : b \in B\} = d(12, 22) = 10$. We can note that for each $a \in A$, the closest point in B that gives the smallest distance will always be $b = 22$. Therefore, we find that $\rho(A, B) = \sup\{d(a, 22) : a \in A\}$. The point $a = 0$ in A maximizes this distance. Therefore $\rho(A, b) = d(0, 22) = |22 - 0| = 22$.

Similarly, we find that $\rho(B, A) = \sup\{d(b, 20) : b \in B\}$, since the point $a = 20$ will give the smallest distance to any point in B. The point $b = 31$ in B maximizes this distance, so we have $\rho(B, A) = d(20, 31) = |20 - 31| = 11$.

We will note that $\rho(B, A)$ and $\rho(A, B)$ are not always equal. It follows that $h(A, B) = \max\{\rho(A, B), \rho(B, A)\} = 22$.

Let us consider another example in $(\mathbb{R}, d)$ of discrete sets where the metric $d(x, y) = |x - y|$.

Example Let $A = \{5n : 0 \leq n \leq 19\}$ and $B = \{p : p < 100, p \text{ prime}\}$. Since we are working with discrete, finite subsets of the real number line, we find that $r(p, A)$ is equal to the minimum distance from any prime number $p \in B$ to a multiple of 5 in the set A. That is, for any $p \in B$,

$$r(p, A) = \min\{|p - 5n| : 0 \leq n \leq 19\}.$$

Note that the minimum distance from any prime number to the nearest multiple of 5 is either 2 or 1. For example, if $p = 17$, then we find that

$$r(17, A) = \min\{|17 - 5n| : 0 \le n \le 19\} = 2, \text{ when } n = 3.$$

If $p = 71$, then

$$r(71, A) = \min\{|71 - 5n| : 0 \le n \le 19\} = 1, \text{ where } n = 14.$$

Therefore we find that, for any point $a \in A$, $\rho(B, A) = \max\{r(b, A) : b \in B\} = 2$. In other words, $\rho(B, A)$ is equal to the maximum of the minimum distances from a prime p in the set B to the multiples of 5 in the set A.

Similarly, we find that $\rho(A, B)$ is equal to the maximum of the minimum distances from each multiple of 5 in the set A to the set of primes in the set B. Thus $r(a, B)$ is equal to the minimum distance from any multiple of 5 in the set A to a prime number $p \in B$. There is no efficient way to do this, except by looking at the distances between each point in A to each point in B. . By looking at the distance between all of the multiples of 5 to a prime number we find that the largest of these minimum distances occurs at the point $50 \in A$ and $47, 53 \in B$. Therefore $h(A, B) = \max\{\rho(A, B), \rho(B, A)\} = \max\{3, 2\} = 3$.

Now consider the following example of r, ρ, and h in the complete metric space $(\mathbb{R}^2, d)$, where $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Example Let A and B be subsets of $\mathbb{R}^2$ defined by $A = \{(x, y) : x^2 + y^2 \le 1\}$ and $B = \{(x, y) : 0 \le x \le 3, 0 \le y \le 1\}$ [see Figure 1].

By definition, $r(x, A) = \inf\{d(x, a) : a \in A\}$. So $r(x, A)$ is the set of all distances from each $x \in B$ to the "closest" point $a \in A$ to x. Note that x will be an ordered pair, as are a and b.

If $b \in A \cap B$, then it is clear that $r(b, A) = 0$. If $b \in B \backslash A$, then $r(b, A)$ is found using the line from b to the origin [see Figure 2]. We find that the point that yields the largest distance is the upper right vertex of the rectangle at the point $(3, 1)$. Therefore, $\rho(B, A)$ is equal to the distance from the point $\left(\dfrac{3}{\sqrt{10}}, \dfrac{1}{\sqrt{10}}\right)$ on the circle to the point $(3, 1)$.

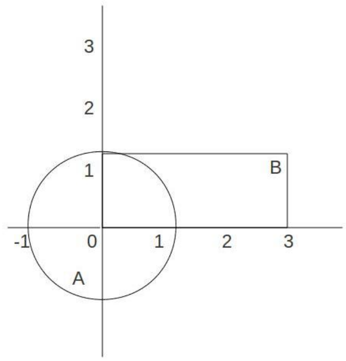

FIGURE 1. Graph of A and B in $(\mathbb{R}^2, d)$.

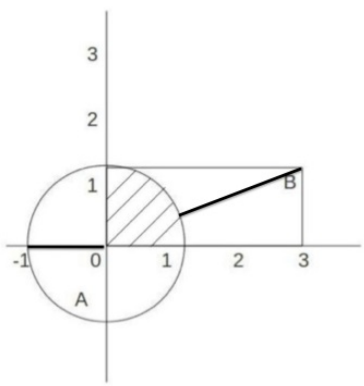

FIGURE 2. Area shaded contains all points that give the infimum distance $r(a, B) = 0$. Additionally, we find $\rho(A, B) = 1$ and $\rho(B, A) = \sqrt{10} - 1$.

Thus,

$$\rho(B, A) = d\left(\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right), (3, 1)\right) = \sqrt{10} - 1.$$

Now we must find $\rho(A, B)$. We find that we can use any of the points in the bottom lower left quadrant on the unit circle. Let us choose the point $(-1, 0)$. Then $\rho(A, B) = d\big((-1, 0), (0, 0)\big) = 1$.

Therefore, the Hausdorff distance is $h(A, B) = \max\{1, \sqrt{10} - 1\} = \sqrt{10} - 1$.

The next example is also in the metric space $(\mathbb{R}^2, d)$ with the same metric d from the previous example. However, this time we will consider two subsets of the plane that do not intersect.

Example Let A and B be sets defined by $A = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and let $B = \{(x, y) : 3 \leq x \leq 5, 0 \leq y \leq 4\}$ [see Figure 3].

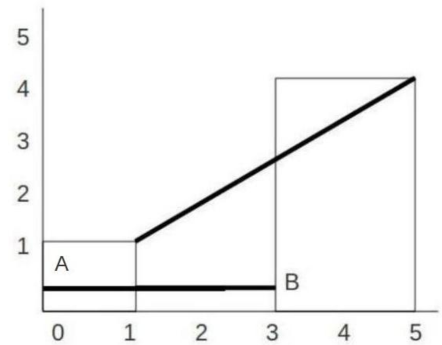

FIGURE 3. The sets A and B in $(\mathbb{R}^2, d)$.

If $(a_1, a_2) \in A$, then $r\big((a_1, a_2), B)\big) = d\big((a_1, a_2), (3, a_2)\big) = 3 - a_1$. Since $0 \leq a_1 \leq 1$, we find that $\rho(A, B) = 3$.

If $(b_1, b_2) \in B$, then $r\big((b_1, b_2), A)\big) = d\big((b_1, b_2), (1, a_2)\big)$, where $0 \leq a_2 \leq 1$, which varies with our choice of (b_1, b_2). We find that the point that maximizes r is $b = (5, 4)$ such that $\rho(B, A) = d\big((5, 4), (1, 1)\big) = 5$.

Therefore, the Hausdorff distance is given by $h(A, B) = \rho(B, A) = 5$.

Now that we have gained a knowledge on how r, ρ, and h work in a few special cases, we prove some basic properties of r and ρ.

Theorem 1. *Let $x \in X$ and let $A, B, C \in \mathcal{K}$.*

(1) *$r(x, A) = 0$ if and only if $x \in A$.*

(2) *$\rho(A, B) = 0$ if and only if $A \subseteq B$.*

(3) *There exists $a_x \in A$ such that $r(x, A) = d(x, a_x)$.*

(4) *There exists $a^* \in A$ and $b^* \in B$ such that $\rho(A, B) = d(a^*, b^*)$.*

(5) *If $A \subseteq B$, then $r(x, B) \leq r(x, A)$.*

(6) *If $B \subseteq C$, then $\rho(A, C) \leq \rho(A, B)$.*

(7) *$\rho(A \cup B, C) = \max\{\rho(A, C), \rho(B, C)\}$.*

(8) *$\rho(A, B) \leq \rho(A, C) + \rho(C, B)$.*

Proof. We first prove Property (1). Suppose $x \in A$. Then the infimum distance is $d(x,a) = 0$, where $a = x$. Now suppose that $r(x,A) = 0$. Then for each positive integer n, there exists $a_n \in A$ such that $d(x,a_n) < \frac{1}{n}$. By definition $\{a_n\}$ converges to x. Since A is compact, it is closed. By Lemma 1 it follows that $x \in A$.

We now prove Property (2). Suppose $A \subseteq B$. Let $a \in A$. Since $a \in B$, by Property (1) then $r(a,B) = 0$. Therefore $\rho(A,B)$ =sup$\{0\} = 0$. To prove the converse, suppose $\rho(A,B) = 0$. Let $a \in A$. Then $0 \leq r(a,B) \leq \rho(A,B) = 0$, and thus by Property (1), we find $a \in B$. It follows that $A \subseteq B$.

To prove Property (3), by definition of an infimum we can let $\{a_n\}$ be a sequence in A such that $d(x,a_n) < r(x,A) + \frac{1}{n}$. We know A is sequentially compact, so there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ that converges to an element $a_x \in A$. Then we find that

$$r(x,A) \leq d(x,a_x) \leq d(x,a_{n_k}) + d(a_{n_k},a_x) \leq r(x,A) + \frac{1}{n_k} + d(a_{n_k},a_x).$$

Since $\lim\limits_{k\to\infty} \big(\frac{1}{n_k} + d(a_{n_k},a_x)\big) = 0$, it follows that $d(x,a_x) = r(x,A)$.

To prove Property (4), by definition of a supremum we can let $\{a_n\}$ be a sequence in A such that $\rho(A,B)$ is the limit of $r(a_n,B)$. By Property (3) there exists a sequence $\{b_n\}$ in B such that $r(a_n,B) = d(a_n,b_n)$. Since A is sequentially compact, there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ that converges to an element $a^* \in A$. Since B is sequentially compact, there exists a subsequence $\{b_{n_{k_j}}\}$ of $\{b_{n_k}\}$ that converges to b^*. Since $\{b_{n_{k_j}}\}$ converges to b^* and $\{a_{n_{k_j}}\}$ converges to a^*, by Result 1, we know that $d(a_{n_{k_j}},b_{n_{k_j}})$ converges to $d(a^*,b^*)$. Therefore, it follows that

$$\rho(A,B) = \lim_{j\to\infty} r(a_{n_{k_j}},B) = \lim_{j\to\infty} r(a_{n_{k_j}},b_{n_{k_j}}) = d(a^*,b^*).$$

Thus it follows that $d(a^*,b^*) = \rho(A,B)$.

Now we will prove Property (5). Suppose $A \subseteq B$ and $x \in X$. Let $a \in A$. Since A is a subset of B, we find that $a \in B$. It follows that

$$d(x,a) \geq \inf\{d(x,b) : b \in B\} = r(x,B).$$

Since this is true for all $a \in A$, we find that $r(x, A) = \inf\{d(x, a) : a \in A\} \geq r(x, B)$.

For Property (6), suppose $B \subseteq C$. By Property (5) then $r(a, C) \leq r(a, B)$ for all $a \in A$. It follows that $\sup\{r(a, C) : a \in A\} \leq \sup\{r(a, B) : a \in A\}$ and thus $\rho(A, C) \leq \rho(A, B)$.

For Property (7), by the definitions of r and ρ we see that

$$\begin{aligned} \rho(A \cup B, C) &= \sup\{r(x, C) : x \in A \cup B\} \\ &= \max\{\sup\{r(x, C) : x \in A\}, \sup\{r(x, C) : x \in B\}\} \\ &= \max\{\rho(A, C), \rho(B, C)\}. \end{aligned}$$

We now turn to Property (8). Property (3) guarantees that for each $a \in A$ there exists $c_a \in C$ such that $r(a, C) = d(a, c_a)$. We then have

$$\begin{aligned} r(a, B) &= \inf\{d(a, b) : b \in B\} \\ &\leq \inf\{d(a, c_a) + d(c_a, b) : b \in B\} \\ &= d(a, c_a) + \inf\{d(c_a, b) : b \in B\} \\ &= r(a, C) + r(c_a, B) \\ &\leq \rho(A, C) + \rho(C, B). \end{aligned}$$

Since $a \in A$ was arbitrary, taking the supremum, we find that

$$\rho(A, B) \leq \rho(A, C) + \rho(C, B).$$

This completes the proof.

□

Note that by Property (4) of Theorem 1, there exist points $a_1 \in A$ and $b_1 \in B$ such that $\rho(A, B) = d(a_1, b_1)$ and alternatively there exist points $a_2 \in A$ and $b_2 \in B$ such that $\rho(B, A) = d(a_2, b_2)$. Since $h(A, B)$ is just the maximum of $\rho(A, B)$ and $\rho(B, A)$, it follows that there exist points $a^* \in A$ and $b^* \in B$ such that $h(A, B) = d(a^*, b^*)$.

The following theorem shows that the Hausdorff distance defines a metric on $\mathcal{K}$.

Theorem 2. *The set $\mathcal{K}$ with the Hausdorff distance h define a metric space $(\mathcal{K}, h)$.*

Proof. To prove that $(\mathcal{K}, h)$ is a metric space, we need to verify the following four properties.

(1) $h(A, B) \geq 0$ for all $A, B \in \mathcal{K}$.
(2) $h(A, B) = 0$ if and only if $A = B$.
(3) $h(A, B) = h(B, A)$ for all $A, B \in \mathcal{K}$.
(4) $h(A, B) \leq h(A, C) + h(C, B)$ for all $A, B, C \in \mathcal{K}$.

To prove the first property, since $\rho(A, B)$ and $\rho(B, A)$ are nonnegative, it follows that $h(A, B) \geq 0$ for all $A, B \in \mathcal{K}$.

For the second property, suppose $A = B$. Therefore $A \subseteq B$ and $B \subseteq A$. By Property (2) of Theorem 1 we find that $\rho(A, B) = 0$ and $\rho(B, A) = 0$, and thus $h(A, B) = 0$. Now suppose $h(A, B) = 0$. This implies $\rho(A, B) = \rho(B, A) = 0$. By Property (2) of Theorem 1, we see that $A \subseteq B$ and $B \subseteq A$ and it follows that $A = B$.

The third property can be proved from the symmetry of the definition since

$$h(A, B) = \max\{\rho(A, B), \rho(B, A)\} = \max\{\rho(B, A), \rho(A, B)\} = h(B, A).$$

The final property follows from the definition of ρ and h and from Property (8) of Theorem 1. We find that

$$\begin{aligned}\rho(A, B) \leq \rho(A, C) + \rho(C, B) &\leq \max\{\rho(A, C), \rho(C, A)\} + \max\{\rho(C, B), \rho(B, C)\} \\ &= h(A, C) + h(C, B).\end{aligned}$$

Similarly,

$$\begin{aligned}\rho(B, A) \leq \rho(B, C) + \rho(C, A) &\leq \max\{\rho(B, C), \rho(C, B)\} + \max\{\rho(C, A), \rho(A, C)\} \\ &= h(C, B) + h(A, C).\end{aligned}$$

Therefore, $h(A, B) = \max\{\rho(A, B), \rho(B, A)\} \leq h(A, C) + h(C, B)$. □

Therefore we know that h defines a metric on $\mathcal{K}$. In the next section, we will look at examples of what this metric space might look like, and then proceed to prove that if the metric space (X, d) is complete, then the metric space $(\mathcal{K}, h)$ is also complete.

Examples of the Hausdorff Metric Space

Given a complete metric space (X, d), we have now constructed a new metric space $(\mathcal{K}, h)$ from the nonempty, compact subsets of X using the Hausdorff metric. Now it remains for us to prove that $(\mathcal{K}, h)$ is also complete. To be a complete metric space, every Cauchy sequence in the space must converge to a point also in the space. Therefore, when we think about the metric space $(\mathcal{K}, h)$, we are choosing a sequence of nonempty, compact sets and showing that this sequence converges to another nonempty, compact set. To better visualize the abstract mathematics we are doing, consider the following two examples that demonstrate metric spaces of nonempty, compact sets with the Hausdorff metric.

Example Let $(\mathbb{R}, d_0)$ be the complete metric space, where d_0 is the discrete metric,

$$d_0(x, y) = \begin{cases} 0, \text{ when } x = y. \\ 1, \text{ when } x \neq y. \end{cases}$$

Since $\mathcal{K}$ is the set of all nonempty, compact subsets of $(\mathbb{R}, d_0)$, we find that $\mathcal{K}$ is the set of all nonempty finite subsets of $\mathbb{R}$. The infinite sets are not in $\mathcal{K}$ because they are not totally bounded and are thus not compact.

Furthermore, we may notice that

$$r(x, B) = \inf\{d_0(x, b) : b \in B\} = d_0(x, b) = \begin{cases} 0, \text{ when } x \in B. \\ 1, \text{ when } x \notin B. \end{cases}$$

Therefore,

$$\rho(A,B)=\sup\{r(a,B):a\in A\}=\begin{cases}0,\text{ when } a\in B.\\[2ex] 1,\text{ when } a\notin B.\end{cases}$$

.

So it follows that

$$h(A,B)=\begin{cases}0,\text{ when } A=B.\\[2ex] 1,\text{ when } A\neq B.\end{cases}$$

Therefore we have a metric space with the set $\mathcal{K}$ of the discrete subsets of $\mathbb{R}$ with the Hausdorff metric as the discrete metric. It is easy to verify that our newly created space is not totally bounded. However, we know all discrete metric spaces are complete, so $(\mathcal{K},h)$ is complete. Therefore, the space $(\mathcal{K},h)$ of finite sets with the discrete metric is an example of our Hausdorff induced metric space $(\mathcal{K},h)$.

To illustrate our notion of completeness, now briefly consider a sequence of nonempty compact sets that converges to the unit circle in $\mathbb{R}^2$. This is an example a converging Cauchy sequence in the Hausdorff induced metric space that converges to a set also in the space.

Example Let $(\mathbb{R}^2,d)$ be the complete metric space where for $x=(x_1,x_2)$ and for $y=(y_1,y_2)$, then $d((x_1,y_1),(x_2,y_2))=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$.

Let $\mathcal{K}$ be the set of all nonempy, compact subsets of $\mathbb{R}^2$, or in other words let $\mathcal{K}$ be the set of all nonempty closed and bounded sets of $\mathbb{R}^2$. As we will later prove, we know that since $(\mathbb{R}^2,d)$ is complete, then the metric space $(\mathcal{K},h)$ is complete. To see an example of a Cauchy sequence in this space that converges to something in the set, let us consider a sequence of sets converging to the unit circle.

For each positive integer k, let $A_k=\{(r,\theta):r=1+\frac{1}{k}\cos(k\theta),0\leq\theta\leq 2\pi\}$, and let A be the unit circle in $\mathbb{R}^2$. It is easy to see that each A_k is in $\mathcal{K}$. Examining the Hausdorff distance between sets, we see that $h(A_k,A)=\frac{1}{k}$.

We can note that as k increases, the sets converge to the unit circle [See figures 6-8]. Therefore, $\{A_k\}$ is an example of a Cauchy sequence that converges to $A \in \mathcal{K}$.

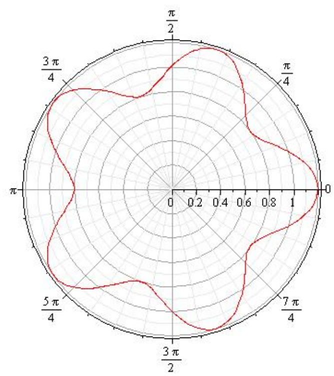

FIGURE 4. The set A_5 defined by $r = 1 + \frac{1}{5}\cos(5\theta)$.

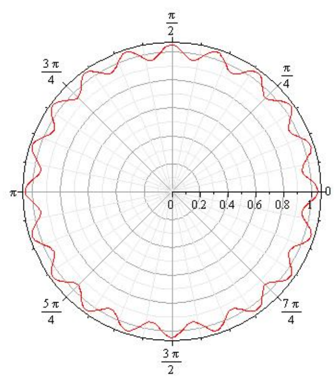

FIGURE 5. The set A_{20} defined by $r = 1 + \frac{1}{20}\cos(20\theta)$.

PROVING THAT THE METRIC $(\mathcal{K}, h)$ IS COMPLETE

As previously stated, to be a complete metric space, every Cauchy sequence in $(\mathcal{K}, h)$ must converge to a point in $\mathcal{K}$. Therefore, in order to prove that the metric space $(\mathcal{K}, h)$ is complete, we will choose an arbitrary Cauchy sequence $\{A_n\}$ in $\mathcal{K}$ and show that it converges to some $A \in \mathcal{K}$. Define A to be the set of all points $x \in X$ such that there is a sequence $\{x_n\}$ that converges to x and satisfies $x_n \in A_n$

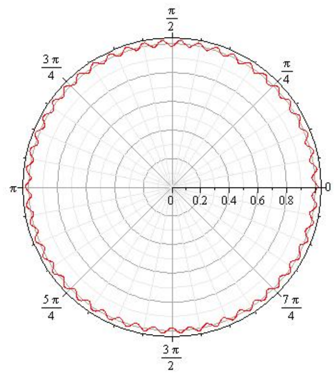

FIGURE 6. The set A_{50} defined by $r = 1 + \frac{1}{50}\cos(50\theta)$.

for all n. We will eventually show that the set A is an appropriate candidate. However, we must begin with some important theorems regarding A.

Given a set $A \in \mathcal{K}$ and a positive number ϵ, we define the set $A + \epsilon$ by $\{x \in X : r(x, A) \leq \epsilon\}$. We need to show that this set is closed for all possible choices of A and ϵ. To do this, we will begin by choosing an arbitrary limit point of the set $A + \epsilon$, and then showing that it is contained in the set.

Proposition 1: $A + \epsilon$ is closed for all possible choices of $A \in \mathcal{K}$ and $\epsilon > 0$.

Proof. Let $A \in \mathcal{K}$ and $\epsilon > 0$. Additionally, let x be a limit point of $A + \epsilon$. Then there exists a sequence $\{x_n\}$ of points in $(A + \epsilon)\backslash\{x\}$ that converges to x. Since $x_n \in A + \epsilon$ for all n, by definition $r(x_n, A) \leq \epsilon$ for all n. Property (3) of Theorem 1 guarantees that for each n there exists $a_n \in A$ such that $r(x_n, A) = d(x_n, a_n)$. Thus $d(x_n, a_n) \leq \epsilon$ for all n. The set A is sequentially compact, so it follows from the definition that each sequence $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ that converges to a point $a \in A$. Since $\{x_n\}$ converges to x, we know that any subsequence of $\{x_n\}$ converges to x. Therefore, the subsequence $\{x_{n_k}\}$ converges to x. By Result 1, then $d(x_{n_k}, a_{n_k})$ converges to $d(x, a)$. Note that $\{x_{n_k}\}$ and $\{a_{n_k}\}$ are subsequences of $\{x_n\}$ and $\{a_n\}$, respectively, so $d(x_{n_k}, a_{n_k}) \leq \epsilon$ for all k. Therefore, we find that $d(x, a) \leq \epsilon$. By the definition of $r(x, A)$, then $r(x, A) \leq \epsilon$, so $x \in A + \epsilon$ by our definition of $A+\epsilon$. Since x was an arbitrary limit point, then $A+\epsilon$ is a closed set since it contains all of its limit points. □

As an example where the set $A+\epsilon$ is not compact, consider the set $\mathbb{R}$ with the discrete metric. Let A be any nonempty finite set and choose $\epsilon > 1$. Then the set $A+\epsilon = \mathbb{R}$ is closed, but not totally bounded and is therefore is not compact.

To show $(\mathcal{K}, h)$ is complete, we will need to show that $A \in \mathcal{K}$, and that $\{A_n\}$ converges to A. By our definition of convergence, we must show that there exists a positive integer N such that $h(A_n, A) < \epsilon$ for all $n \geq N$. However, the following theorem gives us an alternative way of proving convergence.

Theorem 3. *Suppose that $A, B \in \mathcal{K}$ and that $\epsilon > 0$. Then $h(A,B) \leq \epsilon$ if and only if $A \subseteq B+\epsilon$ and $B \subseteq A+\epsilon$.*

Proof. By symmetry, it is sufficient to prove $\rho(B,A) \leq \epsilon$ if and only if $B \subseteq A+\epsilon$. Suppose $B \subseteq A+\epsilon$. By definition of the set $A+\epsilon$, for every $b \in B$ then $r(b,A) \leq \epsilon$. It follows that $\rho(B,A) \leq \epsilon$. Now suppose $\rho(B,A) \leq \epsilon$. Then for every $b \in B$ then $r(b,A) \leq \epsilon$. It follows by definition of the set $A+\epsilon$, that $B \subseteq A+\epsilon$. □

Extension Lemma: Let $\{A_n\}$ be a Cauchy sequence in $\mathcal{K}$ and let $\{n_k\}$ be an increasing sequence of positive integers. If $\{x_{n_k}\}$ is a Cauchy sequence in X for which $x_{n_k} \in A_{n_k}$ for all k, then there exists a Cauchy sequence $\{y_n\}$ in X such that $y_n \in A_n$ for all n and $y_{n_k} = x_{n_k}$ for all k.

Proof. Suppose $\{x_{n_k}\}$ is a Cauchy sequence in X for which $x_{n_k} \in A_{n_k}$ for all k. Define $n_0 = 0$. For each n that satisfies $n_{k-1} < n \leq n_k$, use Property 3 to choose $y_n \in A_n$ such that $r(x_{n_k}, A_n) = d(x_{n_k}, y_n)$. Then we find, using the definitions of ρ and r that

$$d(x_{n_k}, y_n) = r(x_{n_k}, A_n) \leq \rho(A_{n_k}, A_n) \leq h(A_{n_k}, A_n).$$

Note that since $x_{n_k} \in A_{n_k}$, then $d(x_{n_k}, y_{n_k}) = r(x_{n_k}, A_{n_k}) = 0$. It follows that $y_{n_k} = x_{n_k}$ for all k.

Let $\epsilon > 0$. Since $\{x_{n_k}\}$ is a Cauchy sequence in X, there exists a positive integer K such that $d(x_{n_k}, x_{n_j}) < \frac{\epsilon}{3}$ for all $k, j \geq K$. Since $\{A_n\}$ is a Cauchy sequence in $\mathcal{K}$, by definition there exists a positive integer $N \geq n_K$ such that $h(A_n, A_m) < \frac{\epsilon}{3}$ for all $n, m \geq N$. Suppose that $n, m \geq N$. Then there exists integers $j, k \geq K$ such that $n_{k-1} < n \leq n_k$, and $n_{j-1} < m \leq n_j$. Then we find that

$$\begin{aligned}
d(y_n, y_m) &\leq d(y_n, x_{n_k}) + d(x_{n_k}, x_{n_j}) + d(x_{n_j}, y_m) \\
&= r(x_{n_k}, A_n) + d(x_{n_k}, x_{n_j}) + r(x_{n_j}, A_m) \\
&\leq \rho(A_{n_k}, A_n) + d(x_{n_k}, x_{n_j}) + \rho(A_{n_j}, A_m) \\
&\leq h(A_{n_k}, A_n) + d(x_{n_k}, x_{n_j}) + h(A_{n_j}, A_m) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Therefore, by definition and from our earlier set up, $\{y_n\}$ is a Cauchy sequence in X such that $y_n \in A_n$ for all n and $y_{n_k} = x_{n_k}$ for all k. This completes the proof. $\square$

The following Lemma makes use of the Extension Lemma to guarantee that A is closed and nonempty. We will need this fact in proving that A is in $\mathcal{K}$, since we must show that A is a nonempty, compact subset of $\mathcal{K}$. This Lemma gives us that A is closed and nonempty. Since closed and totally bounded sets are compact, it remains to show that A is totally bounded.

Lemma 2: Let $\{A_n\}$ be a sequence in $\mathcal{K}$ and let A be the set of all points $x \in X$ such that there is a sequence $\{x_n\}$ that converges to x and satisfies $x_n \in A_n$ for all n. If $\{A_n\}$ is a Cauchy sequence, then the set A is closed and nonempty.

Proof. We begin by proving that A is nonempty. Since $\{A_n\}$ is a Cauchy sequence, there exists an integer n_1 such that $h(A_m, A_n) < \frac{1}{2^1} = \frac{1}{2}$ for all $m, n \geq n_1$. Similarly, there exists an integer $n_2 > n_1$ such that $h(A_m, A_n) < \frac{1}{2^2} = \frac{1}{4}$ for all $m, n \geq n_2$. Continuing this process we have an increasing sequence $\{n_k\}$ such that $h(A_m, A_n) < \frac{1}{2^k}$ for all $m, n \geq n_k$. Let x_{n_1} be a fixed point in A_{n_1}. By Property 2 of Theorem 2, we can choose $x_{n_2} \in A_{n_2}$ such that $d(x_{n_1}, x_{n_2}) = r(x_{n_1}, A_{n_2})$. Then by definition of r, ρ, and h we find that

$$d(x_{n_1}, x_{n_2}) = r(x_{n_1}, A_{n_2}) \leq \rho(A_{n_1}, A_{n_2}) \leq h(A_{n_1}, A_{n_2}) < \tfrac{1}{2}.$$

Similarly we can choose $x_{n_3} \in A_{n_3}$ such that

$$d(x_{n_2}, x_{n_3}) = r(x_{n_2}, A_{n_3}) \leq \rho(A_{n_2}, A_{n_3}) \leq h(A_{n_2}, A_{n_3}) < \tfrac{1}{4}.$$

Continuing this process we can construct a sequence $\{x_{n_k}\}$ where each $x_{n_k} \in A_{n_k}$ and for all k,

$$d(x_{n_k}, x_{n_{k+1}}) = r(x_{n_k}, A_{n_{k+1}}) \leq \rho(A_{n_k}, A_{n_{k+1}}) \leq h(A_{n_k}, A_{n_{k+1}}) < \tfrac{1}{2^k}.$$

By Result 2 $\{x_{n_k}\}$ is a Cauchy sequence.

Therefore, since $\{x_{n_k}\}$ is a Cauchy sequence and $x_{n_k} \in A_{n_k}$ for all k, by the Extension Lemma there exists a Cauchy sequence $\{y_n\}$ in X such that $y_n \in A_n$ for all n and $y_{n_k} = x_{n_k}$ for all k. Since X is complete, the Cauchy sequence $\{y_n\}$ converges to a point $y \in X$. Since $y_n \in A_n$ for all n, then by definition of the set, $y \in A$. Therefore A is nonempty.

Now we will prove that A is closed. Suppose a is a limit point of A. Then there exists a sequence $a_k \in A \backslash \{a\}$ that converges to a. Since each $a_k \in A$, there exists a sequence $\{y_n\}$ such that $\{y_n\}$ converges to a_k and $y_n \in A_n$ for each n. Consequently, there exists an integer n_1 such that $x_{n_1} \in A_{n_1}$ and $d(x_{n_1}, a_1) < 1$. Similarly, there exists an integer $n_2 > n_1$ and a point $x_{n_2} \in A_{n_2}$ such that $d(x_{n_2}, a_2) < \frac{1}{2}$. Continuing this process we can choose an increasing sequence $\{n_k\}$ of integers such that $d(x_{n_k}, a_k) < \frac{1}{k}$ for all k. Then it follows that

$$d(x_{n_k}, a) \leq d(x_{n_k}, a_k) + d(a_k, a).$$

Note that as we take k to infinity, the distance between $\{x_{n_k}\}$ and a converges to zero, so it follows that $\{x_{n_k}\}$ converges to a. Every convergent sequence is Cauchy, so it follows that $\{x_{n_k}\}$ is a Cauchy sequence for which $x_{n_k} \in A_{n_k}$ for all k. The Extension Lemma guarantees that there exists a Cauchy sequence $\{y_n\}$ in X such that $y_n \in A_n$ for all n and $y_{n_k} = x_{n_k}$. Therefore $a \in A$, so A is closed. $\square$

With the previous lemma, to prove $A \in \mathcal{K}$, it only remains to show that A is totally bounded. The following lemma will allow us to do so.

Lemma 3: Let $\{D_n\}$ be a sequence of totally bounded sets in X and let A be any subset of X. If for each $\epsilon > 0$, there exists a positive integer N such that $A \subseteq D_N + \epsilon$, then A is totally bounded.

Proof. Let $\epsilon > 0$. Choose a positive integer N so that $A \subseteq D_N + \frac{\epsilon}{4}$. Since D_N is totally bounded, by definition we can choose a finite set $\{x_i : 1 \leq i \leq q\}$ where $x_i \in D_N$ such that $D_N \subseteq \bigcup_{i=1}^{q} B_d(x_i, \dfrac{\epsilon}{4})$. By reordering the x_i's, we may assume that $B_d(x_i, \frac{\epsilon}{2}) \cap A \neq \emptyset$ for $1 \leq i \leq p$ and $B_d(x_i, \frac{\epsilon}{2}) \cap A = \emptyset$ for $p < i$. Then for each $1 \leq i \leq p$, let $y_i \in B_d(x_i, \frac{\epsilon}{2}) \cap A$. We claim that $A \subseteq \bigcup_{i=1}^{p} B_d(y_i, \epsilon)$. Let $a \in A$. Then $a \in D_N + \frac{\epsilon}{4}$, so $r(a, D_N) \leq \frac{\epsilon}{4}$. By Theorem 1 Property (3), then there exists $x \in D_N$ such that $d(a, x) = r(a, D_N)$. Then we find that

$$d(a, x_i) \leq d(a, x) + d(x, x_i) \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.$$

So $x \in B_d(x_i, \frac{\epsilon}{2})$ for some $1 \leq i \leq p$. Thus we have $y_i \in B_d(x_i, \frac{\epsilon}{2}) \cap A$ such that $d(x_i, y_i) < \frac{\epsilon}{2}$. It follows that

$$d(a, y_i) \leq d(a, x_i) + d(x_i, y_i) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore since for each $a \in A$ we found y_i for $1 \leq i \leq p$ such that $a \in B_d(y_i, \epsilon)$, then it follows that $A \subseteq \bigcup_{i=1}^{p} B_d(y_i, \epsilon)$. Thus by definition, A is totally bounded. This completes the proof. $\square$

Finally, we have the foundation to prove our final result. Given a complete metric space (X, d), we constructed the metric space $(\mathcal{K}, h)$ from the nonempty compact subsets of X using the Hausdorff metric. After examining important theorems and results, we can now prove our main goal.

Theorem 4. *If (X, d) is complete, then $(\mathcal{K}, h)$ is complete.*

Proof. Let $\{A_n\}$ be a Cauchy sequence in $\mathcal{K}$, and define A to be the set of all points $x \in X$ such that there is a sequence $\{x_n\}$ that converges to x and satisfies $x_n \in A_n$ for all n. We must prove that $A \in \mathcal{K}$ and $\{A_n\}$ converges to A.

By Lemma 2, the set A is closed and nonempty. Let $\epsilon > 0$. Since $\{A_n\}$ is Cauchy then there exists a positive integer N such that $h(A_n, A_m) < \epsilon$ for all $m, n \geq N$. By Theorem 3 then $A_m \subseteq A_n + \epsilon$ for all $m > n \geq N$. Let $a \in A$. Then we want to show $a \in A_n + \epsilon$. Fix $n \geq N$. By definition of the set A, there exists a sequence $\{x_i\}$ such that $x_i \in A_i$ for all i and $\{x_i\}$ converges to a. By Proposition 1 we know that $A_n + \epsilon$ is closed. Since $x_i \in A_n + \epsilon$ for each i, then it follows that $a \in A_n + \epsilon$. This shows that $A \subseteq A_n + \epsilon$. By Lemma 3, the set A is totally bounded. Additionally, we know A is complete, since it is a closed subset of a complete metric space. Since A is nonempty, complete and totally bounded, then A is compact and thus $A \in \mathcal{K}$.

Let $\epsilon > 0$. To show that $\{A_n\}$ converges to $A \in \mathcal{K}$, we need to show that there exists a positive integer N such that $h(A_n, A) < \epsilon$ for all $n \geq N$. To do this, Theorem 3 tells us that we need to show two conditions, that $A \subseteq A_n + \epsilon$ and $A_n \subseteq A + \epsilon$. From the first part of our proof, we know there exists N such that $A \subseteq A_n + \epsilon$ for all $n \geq N$.

To prove $A_n \subseteq A + \epsilon$, let $\epsilon > 0$. Since $\{A_n\}$ is a Cauchy sequence, we can choose a positive integer N such that $h(A_m, A_n) < \frac{\epsilon}{2}$ for all $m, n \geq N$. Since $\{A_n\}$ is a Cauchy sequence in $\mathcal{K}$, there exists a strictly increasing sequence $\{n_i\}$ of positive integers such that $n_1 > N$ and such that $h(A_m, A_n) < \epsilon 2^{-i-1}$ for all $m, n > n_i$. We can use Property (3) of Theorem 1 to get the following:

since $A_n \subseteq A_{n_1} + \frac{\epsilon}{2}$, there exists $x_{n_1} \in A_{n_1}$ such that $d(y, x_{n_1}) \leq \frac{\epsilon}{2}$.
since $A_{n_1} \subseteq A_{n_2} + \frac{\epsilon}{4}$, there exists $x_{n_2} \in A_{n_2}$ such that $d(x_{n_1}, x_{n_2}) \leq \frac{\epsilon}{4}$.
since $A_{n_2} \subseteq A_{n_3} + \frac{\epsilon}{8}$, there exists $x_{n_3} \in A_{n_3}$ such that $d(x_{n_2}, x_{n_3}) \leq \frac{\epsilon}{8}$.

By continuing this process we are able to obtain a sequence $\{x_{n_i}\}$ such that for all positive integers i then $x_{n_i} \in A_{n_i}$ and $d(x_{n_i}, x_{n_{i+1}}) \leq \epsilon 2^{-i-1}$. By Result 2 we find

$\{x_{n_i}\}$ is a Cauchy sequence, so by the Extension Lemma the limit of the sequence a is in A. Additionally we find that

$$\begin{aligned} d(y, x_{n_i}) &\leq d(y, x_{n_1}) + d(x_{n_1}, x_{n_2}) + d(x_{n_2}, x_{n_3}) + \cdots + d(x_{n_{i-1}}, x_{n_i}) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{8} + \cdots + \frac{\epsilon}{2^i} < \epsilon \end{aligned}$$

Since $d(y, x_{n_i}) \leq \epsilon$ for all i, it follows that $d(y, a) \leq \epsilon$ and therefore $y \in A + \epsilon$. Thus we know that there exists N such that $A_n \subseteq A+\epsilon$, so it follows that $h(A_n, A) < \epsilon$ for all $n \geq N$ and thus $\{A_n\}$ converges to $A \in \mathcal{K}$. Therefore, if (X, d) is complete, then $(\mathcal{K}, h)$ is complete.

□

We will now prove that if X is compact, then $\mathcal{K}$ is compact. Note that a metric space is compact if and only if it is complete and totally bounded.

Theorem 5. *If (X, d) is compact, then $(\mathcal{K}, h)$ is compact.*

Proof. By previous result we know that $\mathcal{K}$ is complete. Since we know that a set is compact if and only if it is complete and totally bounded, we must prove that $\mathcal{K}$ is totally bounded. Let $\epsilon > 0$. Since X is totally bounded, there exists a finite set $\{x_i : 1 \leq i \leq n\}$ such that $X \subseteq \bigcup_{i=1}^{n} B_d\big(x_i, \frac{\epsilon}{3}\big)$ and $x_i \in X$ for each i. Let $\{C_k : 1 \leq k \leq 2^p - 1\}$ be the collection of all possible nonempty unions of the closures of these balls. Since X is compact, the closure of each ball is a compact set. Therefore, each C_k is a finite union of compact sets and thus compact, so $C_k \in \mathcal{K}$. We want to show that $\mathcal{K} \subseteq \bigcup_{k=1}^{2^p-1} B_h(C_k, \epsilon)$.

To do this, let $Z \in \mathcal{K}$. Then we want to show that $Z \in B_h(C_k, \epsilon)$ for some k. Choose $S_Z = \{i : Z \cap \overline{B_d(x_i, \epsilon)} \neq \emptyset\}$. Then choose an index j so that $C_j = \bigcup_{i \in S_Z} \overline{B_d\big(x_i, \frac{\epsilon}{3}\big)}$. Since $Z \subseteq C_j$, then by Property (2) of Theorem 1 then we know $\rho(Z, C_j) = 0$. Now let c be an element in C_j. Then there exists some $i \in S_Z$ and $z \in Z$ such that $c, z \in \overline{B_d\big(x_i, \frac{\epsilon}{3}\big)}$. This implies that $r(c, Z) \leq \frac{2}{3}\epsilon$. Since our choice of c was arbitrary, then we find that $\rho(C_j, Z) \leq \frac{2}{3}\epsilon$. Therefore, we find that $h(Z, C_j) = \rho(C_j, Z) < \epsilon$, and thus $Z \subseteq B_h(C_j, \epsilon)$, so $\mathcal{K}$ is totally bounded. Therefore, we have proved that if (X, d) is compact, then $(\mathcal{K}, h)$ is compact.

□

Conclusions

The Hausdorff distance is a measure that assigns a nonnegative real number as the distance between sets. We investigated this distance through several examples. Then, given a metric space (X, d), we found that the Hausdorff distance defines a metric on the space $\mathcal{K}$ of all nonempty compact subsets of X. We explored some of the nice qualities of this metric. Most importantly, that if (X, d) is complete, then $(\mathcal{K}, h)$ is complete. In addition, we proved that if (X, d) is compact, then $(\mathcal{K}, h)$ is totallycompact, which is a truly remarkable result.

References

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